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Some Standard Components of Sporadic Type

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1. INTRODUCTION

In this paper we present a method for proving the nonexistence of finite simple groups having a standard component of a certain type. This method relies heavily on the B-Conjecture and the work of Aschbacher, Seitz, and others on standard component problems for Chevalley groups of characteristic 2.

We shall develop some basic consequences of the B-Conjecture in Section 2 and apply these in Section 3 to obtain an easy classification algorithm. We shall apply this algorithm in the remaining sections to prove the following theorems.

THEOREM 1.1. *Let G be a finite group with $O(G) = \langle 1 \rangle$ having a standard subgroup L with $|Z(L)|$ odd and $L/Z(L)$ isomorphic to one of the following groups*:

(1) *Suzuki's sporadic simple group, Suz.*

(2) *A simple group OS of O'Nan–Sims type.*

(3) *A simple group of F_5 type.*

(4) *Rudvalis' simple group, Ru.*

(5) *Conway's group, .1.*

Then $A \subseteq K_1K_2 \trianglelefteq G$ with $K_i \cong A$ and $K_i \trianglelefteq K_1K_2$ for $i = 1$ or 2.

Remark. If the hypothesis that $|Z(A)|$ is odd replaced by the hypothesis that $|Z(A)|$ is even and the other hypothesis of Theorem 1.1 remain unchanged then it has been proved in [10] and [11] that $A \trianglelefteq G$.

For our next result we need a hypothesis.

HYPOTHESIS 1.2. Let G be a finite group with $F^*(G)$ quasi-simple having a subgroup K with $|Z(K)|$ even, with $KZ(G)/Z(G)$ standard in $\bar{G} = G/Z(G)$ and with $K/Z(G)$ isomorphic to one of the following groups:

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(1) $U_6(2)$.

(2) Fischer's simple group, $M(22)$.

(3) ${}^2E_6(2)$.

(4) A simple group of F_2 type.

Then one of the following holds:

(1) $K/Z(K) \simeq U_6(2)$ and $F^*(\bar{G}) \simeq M(22)$.

(2) $K/Z(K) \simeq M(22)$ and $\bar{G}$ is isomorphic to Fischer's group $M(23)$ or $M(24)'$.

(3) $K/Z(K) \simeq {}^2E_6(2)$ and $\bar{G}$ is of F_2 type.

(4) $K/Z(K)$ is of F_2 type and G is of F_1 type.

Here $M(24)'$ is the commutator subgroup of Fischer's largest 3-transposition group. F_2 is the so-called baby monster and F_1 the so-called monster.

THEOREM 1.3. *Let G be a finite group with $O(G) = \langle 1 \rangle$ having a standard subgroup L with $|Z(L)|$ odd and $L/Z(L)$ isomorphic to one of the following groups*:

(1) $M(22)$ *or* $M(23)$.

(2) $M(24)'$.

(3) *A group of F_2 type.*

(4) *A group of F_1 type.*

Suppose that Hypothesis 1.2 *holds in all sections of G. Then one of the following holds*:

(1) $L \subseteq K_1K_2 \trianglelefteq G$ *with* $K_i \simeq L$ *and* $K_i \trianglelefteq K_1K_2$ *for* $i = 1$ *or* 2.

(2) $L \simeq M(23)$ *and* $F^*(G)$ *is of* $M(24)'$ *type.*

The phrase "H is of H_1 type" means here in general that H is a simple group and Aut H has the same centralizers of involutions as the (predicted) group H_1. The precise meaning of the phrase will be specified in each section. The phrase is used because in some cases the uniqueness (and even existence) of a simple group with the indicated properties is unknown as of the writing of this paper.

We remark that once Hypothesis 1.2 is proved, the standard form problem will be solved for all sporadic simple groups known today, except possibly .2 and F_3.

In the course of proving Theorem 1.1, we also obtain the following result.

THEOREM 1.4. *Let G be a finite group with $O(G) = \langle 1 \rangle$ having a standard subgroup K with $|Z(K)| = 2$, $K/Z(K) \simeq G_2(4)$ and Sylow 2-subgroups of $C_G(K)$ cyclic. Then $K \trianglelefteq G$.*

We note that this result completes the classification of finite groups with a standard subgroup K such that $K/Z(K) \cong G_2(2^n)$. We are assuming here and throughout the paper that groups of Conway type in the sense of Aschbacher and Seitz [6] have been identified as isomorphic to $.1$. We have been so advised by M. Aschbacher [34].

2. Embeddings of Components

A *component* of a group H is a subnormal quasi-simple subgroup of H. $I(H)$ is the set of all involutions of H and $\mathscr{L}(G)$ is the set of all components of 2-local subgroups of G. $L(H)$ is the product of all components of H. $L(H)$ is characteristic in H and $L(H)/Z(L(H))$ is a direct product of simple groups.

We shall assume that all finite groups satisfy the $B(G)$ Property.

$B(G)$ Property. Let G be a finite group with $O(G) = \langle 1 \rangle$. Let t be an involution of G, L a perfect subnormal subgroup of $C_G(t)$ with $L/O(L)$ quasi-simple. Then L is a component of $C_G(t)$.

The basic theorem on embeddings of components is the L-Balance Theorem of Gorenstein and Walter [14, Sects. 3, 4], which may be stated as follows under the assumption of the $B(G)$ Property.

THEOREM 2.1 (L-Balance). *Let G be a finite group with $O(G) = \langle 1 \rangle$. Let S and T be 2-subgroups of G with $[S, T] \subseteq S \cap T$. Let K be a component of $C_G(S)$. Then*

(1) $L(C_K(T)) \subseteq L(C_G(T))$.

(2) *If $T \subseteq C_G(K)$, there exists a component M of $C_G(T)$ with K a component of $C_{\langle M^S \rangle}(S)$. Either $\langle M^S \rangle$ is quasi-simple or $\langle K^{(L(C_G(T))} \rangle$ is a product $K_1 K_2 \cdots K_r$ of components of $C_G(T)$ with K isomorphic to a homomorphic image of K_i for each i.*

The L-Balance Theorem permits us to define a relation on $\mathscr{L}(G)$ as follows. We say that $K > L$ if there exists a pair (S, T) of non-identity 2-subgroups of G with $[S, T] \subseteq S \cap T$ and with $K \lhd\lhd C_G(S)$, $L \lhd\lhd C_G(T)$ and $K \subseteq \langle L^S \rangle$. We extend $<$ to a transitive relation $\leqslant$ on $\mathscr{L}(G)$ and say that K is *maximal* in $\mathscr{L}(G)$ if $K \leqslant L$ implies that $K/Z(K) \cong L/Z(L)$. We let $\mathscr{L}^*(G)$ denote the set of maximal elements in $\mathscr{L}(G)$.

We say that K is *standard* in G if $[K, K^g] \neq 1$ for all $g \in G$ and $| C_G(K) \cap C_G(K)^g |$ is odd for all $g \in G - N_G(K)$. A major consequence of the $B(G)$ Property is the Component Theorem of Aschbacher [2, Theorem 1].

THEOREM 2.2 (Component Theorem). *Let G be a finite group with $O(G) = \langle 1 \rangle$. Suppose that $A \in \mathscr{L}^*(G)$. Then one of the following holds:*

(1) *$A \trianglelefteq G$ or $A \subseteq K_1 * K_2 \trianglelefteq G$ with $K_i/Z(K_i) \cong A/Z(A)$.*

(2) *A is standard in G.*

(3) *A has 2-rank* 1.

We shall make heavy use in the sequel of the following corollary to the Component Theorem.

COROLLARY 2.3. *Let G be a finite group with $F^*(G)$ quasi-simple. Let T_0 be a 2-subgroup of G and K a component of $C_G(T_0)$ with $K/Z(K)$* not *isomorphic to A_7 or to $PSL(2, q)$ for any odd q. Then there exists a chain*

$$K = L_0, L_1, L_2, \ldots, L_{n-1}, L_n = F^*(G)$$

satisfying:

(1) *If $L_i = L_j$, then $i = j$.*

(2) *L_i is a component of $C_G(T_i)$ for some 2-subgroup T_i of G.*

(3) *There is a 2-element s_i of $N_G(T_i)$ with $s_i^2 \in T_i$ and L_{i-1} a component of $C_G(\langle T_i, s_i\rangle)$.*

(4) $L_{i-1} \subseteq L_i(L_i)^{s_i} = \langle (L_{i-1})^{L(C_G(T_i))}\rangle$.

(5) *One of the following holds for each i*:

(a) $L_i \neq (L_i)^{s_i}$; $L_i/Z(L_i) \cong L_{i-1}/Z(L_{i-1})$.

(b) $L_i = (L_i)^{s_i}$; *$L_{i-1}Z(L_i)/Z(L_i)$ is standard in $\langle L_i, s_i\rangle/Z(L_i)$.*

Proof. A slightly stronger result is proved as Corollary 1.4 in [11].

We remark that the brunt of the difficulty in the proof lies in the fact that it seems to be impossible to rule out the existence of components K and L with $K \neq L$ and $K < L < K$. However the Corollary guarantees the existence of *some* chain through K which never returns to K.

We shall also repeatedly make use of the following general results on the centralizers of standard components.

THEOREM 2.4. *Let G be a finite group with $F^*(G)$ simple. Let A be a standard subgroup of G.*

(1) (Aschbacher [3]): *If $C_G(A)$ has generalized quaternion Sylow 2-subgroups, then $F^*(G)$ is a Chevalley group over a field of odd order.*

(2) (Finkelstein [10]): *If $m_2(A) > 1$ and $C_G(A)$ has cyclic Sylow 2-subgroups of order at least* 4, *then* Aut *A has an involution t and a normal subgroup V of $C_{\mathrm{Aut}\,A}(t)/O(C_{\mathrm{Aut}\,A}(t))$ with either $V \cong \mathbb{Z}_4$ or $V \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.*

(3) (Aschbacher–Seitz [6, 34]): *Suppose that $A/Z(A)$ is isomorphic to an alternating group or a Chevalley group or one of the first* 25 *sporadic simple groups. Suppose also that $m_2(C_G(A)) \geq 2$. Then a Sylow 2-subgroup of $C_G(A)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Either $A \cong A_n$ for some $n \geq 5$ or one of the following holds*:

(a) $A \cong L_3(4), F^*(G) \cong Suz$.

(b) $A \cong E_4 \cdot L_3(4), F^*(G) \cong He$, *Held's simple group.*

(c) $A \cong Sz(8), F^*(G) \cong Ru$.

(d) $A \cong G_2(4), F^*(G) \cong .1$.

We shall also make use of the well-known Thompson Transfer Lemma and Glauberman's Z^*-Theorem. Other results and properties of groups needed in only one section will be listed in the appropriate section. Our notation will be standard.

3. The General Method

In this section we shall examine triples $(K, \mathscr{L}, G)$ satisfying the following conditions:

Hypothesis 3.1. (1) G is a finite group with $F^*(G)$ simple. K is a quasi-simple group with $m_2(K) \geqslant 3$. $\mathscr{L}$ is a finite set of quasi-simple groups with $Z(L) = O(L)$ for each $L \in \mathscr{L}$.

(2) If $L_1 \in \mathscr{L}(G)$ with L_1 isomorphic to a member of $\mathscr{L}$, then $L_1 \in \mathscr{L}^*(G)$ and $C_G(L_1)$ has a cyclic Sylow 2-subgroup.

(3) Suppose that $K_1 \in \mathscr{L}(G)$ and $Z_1 \subseteq Z(K_1)$ with $K_1/Z_1 \cong K$. Then $K_1 \notin \mathscr{L}^*(G_1)$ for $O^2(G) \subseteq G_1 \subseteq G$. If $K_1 \lll L_1 \in \mathscr{L}(G)$, then either there exists $X_1 \subseteq Z(L_1)$ with $L_1/X_1 \cong K_1$ or L_1 is isomorphic to a member of $\mathscr{L}$ and $K_1/O(K_1) \cong K$.

(4) Suppose that $K_1 < L_1 \in \mathscr{L}^*(G)$ and $K_1/O(K_1) \cong K$.

(a) There exists a 2-local subgroup N of L_1 with the properties:

(i) $NZ(K_1)/Z(K_1)$ is not isomorphic to a section of $N_G(L_1) \cap C_G(N)$.

(ii) There is no involutory automorphism α of a central extension K_2 of K_1 such that $C_{\langle K_2, \alpha\rangle}(\alpha)$ is isomorphic to a subgroup of $N_G(L_1) \cap C_G(N)$.

(b) K_1 is standard in L_1 and $| C_{\mathrm{Aut}\, L_1}(K_1)|_2 \leqslant 4$.

We shall call any group L_1 which is isomorphic to a member of $\mathscr{L}$, a *group of type* $\mathscr{L}$. We remark that Condition 2 will hold in a minimal counterexample to the classification of groups with a standard subgroup of type $\mathscr{L}$ whenever the groups in $\mathscr{L}$ are of "known" type. Condition 3 amounts to the assumption that all standard form problems for K and the descendants of K *not* of type $\mathscr{L}$ have already been solved. Condition 4 is a fairly mild restriction on the structure of K and groups of type $\mathscr{L}$. If $| Z(K_1)|$ is even, we may take $N = K_1$ in (4a).

DEFINITION 3.2. A *K-chain of length m* in G is a chain $K_1 < K_2 < \cdots < K_m < K_{m+1} = G$ such that

(1) $K_i \in \mathscr{L}(G)$ for $1 \leqslant i \leqslant m$.

(2) Conditions (1)–(5) of Corollary 2.3 hold for the chain.

(3) $K_1 \cong K$.

Let n be the maximum length of a K-chain in G. A *K-extremal pair* is a pair (K_{n-1}, K_n) from a K-chain of length n.

LEMMA 3.3. *Suppose that (K_{n-1}, K_n) is a K-extremal pair in a group G satisfying Hypothesis* 3.1 *for K, G and some set $\mathscr{L}$. Then*

(1) *$K_{n-1}/O(K_{n-1}) \cong K$ and K_n is isomorphic to a member of $\mathscr{L}$.*

(2) *If $K_{n-1} < \mathscr{L}_n \in \mathscr{L}(G)$ and $K_{n-1} \not\cong L_n$, then L_n is of type $\mathscr{L}$ and $L_n \in \mathscr{L}^*(G)$.*

Proof. We are given a chain

$$K_1 < K_2 < \cdots < K_{n-1} < K_n < K_{n+1} = G$$

with $K_1 \cong K$. By 3.1(3) we have for all i, $1 \leqslant i \leqslant n$, that either $K_i/Z_i \cong K$ for some $Z_i \subseteq Z(K_i)$ or K_i is of type $\mathscr{L}$ and $K_{i-1}/O(K_{i-1}) \cong K$. By the definition of a K-chain and the fact that $F^*(G)$ is simple, K_n is standard in G. Thus by 3.1(3), K_n is of type $\mathscr{L}$ and $K_{n-1}/O(K_{n-1}) \cong K$. Thus (1) holds.

Now suppose that $K_{n-1} < L_n \in \mathscr{L}(G)$ and $K_{n-1} \not\cong L_n$. If L_n is of type $\mathscr{L}$, then by 3.1(4), L_n is standard in G. Thus we may assume that K is isomorphic to a homomorphic image of L_n. Applying Lemma 2.3 to L_n, we can find a chain

$$L_n < L_{n+1} < \cdots < L_r = G$$

satisfying conditions (1)–(5) of Lemma 2.3. As K is isomorphic to a homomorphic image of L_n, we have $n + 1 < r$. Thus the composite chain

$$K_1 < K_2 < \cdots < K_{n-1} < L_n < L_{n+1} < \cdots < L_r = G$$

has length $r - 1 > n$. Moreover this chain satisfies every condition of Lemma 2.3 except possibly condition (1). Thus by the maximal choice of n, condition (1) does fail for this chain. As $K_i \neq K_j$ for $i \neq j$ and $L_i \neq L_j$ for $i \neq j$, we must have $K_i = L_j$ for some i, j. If $i = n - 1$, then $j \geqslant n + 1$. As $|K_{n-1}| \leqslant |L_{j-1}| \leqslant |L_j| = |K_{n-1}|$, equality holds. If $i < n - 1$, then $|K_i| \leqslant |K_{n-1}| \leqslant |L_j| = |K_i|$ and again, equality holds. Thus either L_{j-1}, K_{n-1} or K_{n-2}, K_{n-1} are successive terms in our composite chain with both terms isomorphic to K.

Thus we are led to consider a chain $J < K < L$ with K and L satisfying Hypothesis 3.1, with $J \cong K$ and with $T \neq \langle 1 \rangle$ a 2-subgroup of $C_G(K)$, $w \in N_G(T)$ with $w^2 = 1$ and

$$K \times K^w \trianglelefteq L(C_G(T)), \qquad J = C_{K \times K^w}(w).$$

Let $t \in I(C_G(L))$ and let N be a 2-local subgroup of K satisfying Hypothesis 3.1(4). Let $Q = O_2(N)$, $M = \langle (K^w)^{L(C_G(Q))} \rangle$. By the L-Balance Theorem and 3.1(3), either M is of type $\mathscr{L}$ or M is a product of components each isomorphic to a central extension of K.

Suppose that M is of type $\mathscr{L}$. Then $N_G(M) \cap C_G(K^w)$ contains N. This contradicts 3.1(4a). Thus M is a product of components each isomorphic to a central extension of K. Moreover, if $n \in N$ normalizes some component of M, then n centralizes that component. It follows that $K^* = \langle (K^w)^{L(C_G(N))} \rangle \subseteq C_M(N)$ is a product of components of $L(C_G(N))$ each isomorphic to a central extension of K. If t permutes two components of K^*, then $N_G(L) \cap C_G(K)$ has a section isomorphic to $NZ(K)/Z(K)$, contradicting 3.1(4a). Thus t normalizes each component of K^*. But if K_2 is such a component, then $C_{\langle K_2, t \rangle}(t) \subseteq N_G(L) \cap C_G(K)$, violating 3.1(4a). This proves Lemma 3.3.

Notation 3.4. We fix a K-extremal pair (K, L) with $|C_G(L)|_2$ maximal. Let $\langle t_0 \rangle \in \mathrm{Syl}_2(C_G(L))$, $t \in I(\langle t_0 \rangle)$ and $u \in I(C_G(\langle K, t \rangle) - C_G(L))$ If possible, pick $u \in L$.

For $s \in I(G)$, let $K(C_G(s))$ denote the product of all normal subgroups of $C_G(s)$ which are isomorphic to K.

We consider conditions which will guarantee the validity of the following key hypothesis.

HYPOTHESIS 3.5. Hypothesis 3.1 holds and $K = K(C_G(u))$.

LEMMA 3.6. *Assume Hypothesis* 3.1. *Either of the following conditions will guarantee that* $K = K(C_G(u))$:

(1) $u \in K$.

(2) (a) *If* $L_1 \in \mathscr{L}(G)$ *is of type* $\mathscr{L}$, *then* $|\mathrm{Aut}\, L_1 : L_1| \leqslant 2 = |C_G(L_1)|_2$.

(b) $u \in C_G(\langle u, t \rangle)'$.

Proof. Suppose first that $K \ntrianglelefteq L(C_G(u))$. Then by Lemma 3.3(2), $K \subseteq L_1 \trianglelefteq C_G(u)$ with L_1 of type $\mathscr{L}$. As $|Z(L_1)|$ is odd by 3.1, $u \notin L_1$. Thus $u \notin K$, so case (2) holds. As $|C_G(L_1)|_2 = 2$, $\langle u \rangle \in \mathrm{Syl}_2(C_G(L_1))$. As $|\mathrm{Aut}\, L_1 : L_1| \leqslant 2$, a Sylow 2-subgroup of $C_G(u)/L_1 C_G(\langle L_1, u \rangle) = \bar{C}$ has order at most 4 and $\bar{u}$ is central in $\bar{C}$. Thus $\bar{C}$ has a normal 2-complement and $u \notin C_G(u)'$. But $u \in C_G(\langle u, t \rangle)'$ by hypothesis, a contradiction.

Thus $K \trianglelefteq L(C_G(u))$. Write $K(C_G(u)) = KK_1$ with $K_1 \trianglelefteq L(C_G(u))$, $K \cap$

$K_1 \subseteq Z(K)$. As t normalizes $K_1 \subseteq C_G(K)$, the argument of Lemma 3.3 shows that $K_1 \subseteq Z(K)$, i.e., $K = K(C_G(u))$.

DEFINITION 3.7. $\mathscr{V}$ is the set of all pairs (u, uv) satisfying:

(1) $v \in I(\langle K, u\rangle - Z(K))$

(2) $|C_K(v)|_2$ is maximal subject to (1)

(3) $uv \in u^L$.

Our aim is to establish the following condition.

HYPOTHESIS 3.8. Hypothesis 3.5 holds and

(1) $\mathscr{V}$ is invariant under $C_G(u)$.

(2) $C_G(u) \subseteq N_G(L)$.

LEMMA 3.9. *Assume Hypothesis* 3.5. *Let* $R \in \mathrm{Syl}_2(K)$. *The following conditions suffice to verify Hypothesis* 3.8:

(1) $|u^L \cap V| \geq 2$ *for some* 4-*subgroup* $\langle u, v\rangle$ *with* $v \in \langle K, u\rangle - Z(K)$ *and* $|C_K(v)|_2$ *maximal subject to this, i.e.,* $\mathscr{V} \neq \varnothing$.

(2) $C_L(u)$ *contains all* (Aut K)-*fusion of* 4-*subgroups* $\langle u, v\rangle$ *with* $v \in \langle K, u\rangle - Z(K)$ *and* $|C_K(v)|_2$ *maximal subject to this, i.e.,* $\mathscr{V}$ *is* (Aut K)-*invariant.*

(3) $L = \langle K(C_G(u)), K(C_G(uv))\rangle$ *for all* $(u, uv) \in \mathscr{V}$.

Proof. Suppose that $(u, uv) \in \mathscr{V}$, $g \in C_G(u)$ and $(u, uv^g) \notin \mathscr{V}$. Since $K \trianglelefteq C_G(u)$, $v^g \in I(K - Z(K))$ and $|C_K(v^g)|_2 = |C_K(v)|_2$. But g induces an element of Aut K on K, whence $(u, uv) \in \mathscr{V}$. Thus Hypothesis 3.8(1) holds.

Now let $g \in C_G(u)$. By Condition 3,

$$\begin{aligned} L^g &= \langle K(C_G(u))^g, K(C_G(uv))^g\rangle \\ &= \langle K(C_G(u)), K(C_G(uv^g))\rangle = L, \end{aligned}$$

since $(u, uv^g) \in \mathscr{V}$. Thus Hypothesis 3.8(2) holds as well.

LEMMA 3.10. *Let* $S \in \mathrm{Syl}_2(C_L(u))$, $R = S \cap K$. *Condition* 1 *of Lemma* 3.9 *is a consequence of any one of the following conditions*:

(1) $I(R) \subseteq u^L$.

(2) $S = R \times \langle u\rangle$ *and* $v^K \cap Z(R) \neq \varnothing$ *for all* $v \in I(R)$.

(3) $S \notin \mathrm{Syl}_2(L)$, $Z(S) \subseteq \langle R, u\rangle$ *and* $Z(S) \cap Z(K) = \langle u\rangle$.

(4) $S \in \mathrm{Syl}_2(N_L(K))$, $S \notin \mathrm{Syl}_2(L)$, $Z(S) \subseteq R \times C_S(K)$ *and* $N_G(R) \cap N_G(K)$ *is transitive on* $Z(C_S(K))^{\#}$.

Proof. In case (1), clearly $\mathscr{V} \neq \varnothing$. In case (2), Glauberman's Z^*-Theorem guarantees that $\mathscr{V} \neq \varnothing$. In case (3) or (4), let $S < T \in \mathrm{Syl}_2(L)$ and let $w \in N_T(S) - S$. Then $u \neq u^w \in \langle Z(R), C_S(K)\rangle$. In case (3), as $Z(S) \cap Z(K) = \langle u\rangle$, $uu^w \in \langle K, u\rangle - Z(K)$ whence $(u, u^w) \in \mathscr{V}$. In case (4), as K is standard in L, $C_S(K)^w \cap C_S(K) = \langle 1\rangle$. Thus $u^w \notin Z(C_S(K))$. If $u^w \in Z(R)$, then $(u, u^w) \in \mathscr{V}$. If $u^w = z_1u_1$ with $Z_1 \in Z(R)^\#$, $u_1 \in Z(C_S(K))^\#$, let $h \in N_G(R) \cap N_G(K)$ with $u_1{}^h = u$. Then $(z_1u_1)^h = z_1{}^h u \in \langle Z(R), u\rangle$ and $(u, u^{wh}) \in \mathscr{V}$.

LEMMA 3.11. *Condition* 3 *of Lemma* 3.9 *is a consequence of the following assumptions*:

(1) *If* $K_1/Z_1 \simeq K$ *for some* $Z_1 \subseteq Z(K_1)$ *and* $K_1 \subseteq L(C_L(W))$ *for some* $w \in I(N_G(L))$, *then* K_1 *is standard in* $\langle L, w\rangle$.

(2) *Suppose* L_1 *is a simple group which is isomorphic to a section of* L. *Suppose also that* $\mathrm{Aut}\, L_1$ *has a standard subgroup* $K_1 \simeq K$ *with* $m_2(C_{\mathrm{Aut}\, L_1}(K_1)) \geqslant m_2(C_L(K))$. *Then* $L_1 \simeq L$.

Proof. Let $(u, uv) \in \mathscr{V}$, $K_1 = K(C_G(uv))$ and $H = \langle K, K_1\rangle$. If $K_1 \subseteq N_G(K)$, then $K_1 \subseteq K(C_G(u)) = K$, which is not the case. Thus $K \ntrianglelefteq H$. Now K is standard in $\langle H, u\rangle$. Thus either $\langle K^H\rangle/O(\langle K^H\rangle) = L_1$ is simple and K is standard in $\langle L, u\rangle$ or $\langle K^H\rangle = L_1L_1{}^u$ with $L_1/Z_1 \simeq K$ for some $Z_1 \subseteq Z(L)$. In the former case, $L_1 \simeq L$ by (2). Thus $H = L_1 = L$. In the latter case L_1 is standard in L by (1), contradicting $[L_1, L_1{}^u] = 1$.

We now consider some conditions which in conjunction with Hypothesis 3.8 will yield a contradiction.

LEMMA 3.12. *Assume Hypothesis* 3.8 *and Condition* 1 *below. Either of Condition* (2) *or* (3) *will yield a contradiction*:

(1) *Let* $u_1 \in I(N_G(L))$ *with* $O^{2,2'}(C_L(u_1))$ *isomorphic to a subgroup of* $C_L(u)$. *Let* $S_1 \in \mathrm{Syl}_2(C_G(\langle u_1, t\rangle))$. *Then* $\Omega_1(S_1 \cap Z(C_G(\langle u_1, t\rangle))) = \langle u_1, t\rangle$. *Also* $|\,\mathrm{Aut}\, L : L\,| \leqslant 2$.

(2) $L = \mathrm{Aut}\, L$ *and* $w \in C_L(w)'$ *for all* $w \in I(L)$. *Also* $I(C_L(K)) \subseteq u^L$.

(3) $t^G \cap C_G(\langle u, t\rangle) \neq \{t\}$.

Proof. Let $T \in \mathrm{Syl}_2(C_G(t))$. Suppose that $t^g \in C_G(\langle u, t\rangle) - \{t\}$. Let $u_1 = u^{g^{-1}} \in N_G(L)$. Then

$$O^{2,2'}(C_L(u_1)) \subseteq O^{2,2'}(C_G(u_1)) \subseteq O^{2,2'}(N_G(L^{g^{-1}})) \subseteq L^{g^{-1}}.$$

Thus $O^{2,2'}(C_L(u_1))$ is a subgroup of $C_{Lg^{-1}}(u^{g^{-1}})$. Let $S_1 \in \mathrm{Syl}_2(C_G(\langle u_1, t\rangle))$ with $\langle u, t, t^{g^{-1}}\rangle \subseteq S_1$. Then by Condition 1, $\langle u_1, t\rangle = \Omega_1(S_1 \cap Z(C_G(\langle u_1, t\rangle)))$. As $t^{g^{-1}} \in \Omega_1(S_1 \cap Z(C_G(\langle u_1, t\rangle)))$, we have $t^{g^{-1}} = u_1t$. Thus $t = ut^g$. Hence $t^G \cap C_G(\langle u, t\rangle) = \{t, ut\}$. However, as $u \notin Z(L)$, there exists $h \in L$ with

$u^h \in C_L(u) - \{u\}$, by Glauberman's Z^*-Theorem. But then $u^h t = (ut)^h \in (t^G \cap C_G(\langle u, t\rangle)) - \{t, ut\}$, a contradiction.

Thus $t^G \cap C_G(\langle u, t\rangle) = \{t\}$. Suppose that $T \subset T_1 \in \mathrm{Syl}_2(G)$. Let $x \in N_{T_1}(T) - T$. Then $t^x \in Z(T) - \{t\} \subseteq C_G(\langle u, t\rangle) - \{t\}$, a contradiction. Thus $T \in \mathrm{Syl}_2(G)$. Also Condition 2 holds. Thus $T = T_1 \times Q$ where $T_1 = T \cap L$ and Q is cyclic. Moreover $t \notin C_G(t)'$. Thus $t^G \cap T_1 = \varnothing$. It follows by the Thompson Transfer Lemma that $T_1 \in \mathrm{Syl}_2(O^2(G))$. Now $C_{T_1}(K)$ contains a Sylow 2-subgroup of $C_G(K)$ and $I(C_{T_1}(K)) \subseteq u^L$. Thus K is standard in $O^2(G)$, contrary to Hypothesis 3.1(3).

Lemma 3.13. *Condition* (3) *of Lemma* 3.12 *holds if* $t \notin Z^*(G)$ *and for all* $w \in I(\mathrm{Aut}\, L)$, $u^L \cap C_L(w) = \varnothing$.

Proof. Let $T \in \mathrm{Syl}_2(C_G(t))$. If $t \notin Z^*(G)$, then by Glauberman's Z^*-Theorem, there exists $t_1 \in (t^G \cap T) - \{t\}$. As $T \cap \Omega_1(C_G(L)) = \langle t\rangle$, t_1 induces an involutory automorphism, w, on L. Thus $u^h \in C_L(t_1)$ for some $h \in L$. Then $t_1^{h^{-1}} \in (t^G \cap C_G(\langle u, t\rangle)) - \{t\}$.

It follows from Lemmas 3.12 and 3.13 that if the necessary properties of K and L are verified, we can contradict the existence of G. In the ensuing sections we shall illustrate this method for several sporadic simple groups.

4. The Method Applied

In this section we prove the following two results.

Theorem 4.1. *Let G be a finite group with $O(G) = \langle 1\rangle$ having a standard subgroup L with $|Z(L)|$ odd and $L/Z(L)$ isomorphic to one of the following groups*:

(1) *Suzuki's group, Suz.*

(2) *A group of O'Nan–Sims type.*

(3) *A group of F_5 type.*

(4) *A proper covering group of $G_2(4)$.*

(5) *Conway's group .1.*

Then $L \subseteq L_1L_2 \trianglelefteq G$ with $L_i \cong L$ and $L_i \trianglelefteq L_1L_2$ for $i = 1, 2$.

Theorem 4.2. *Let G be a finite group with $O(G) = \langle 1\rangle$ having a standard subgroup L of F_1 or F_2 type. Assume that if H is a simple section of G having a standard subgroup K with $|Z(K)|$ even and with $K/Z(K) \cong {}^2E_6(2)$ or $K/Z(K)$ of F_2 type, then H is of F_2 or F_1 type respectively. Then $L \subseteq L_1L_2 \trianglelefteq G$ with $L_1 \cong L$ and $L_i \trianglelefteq L_1L_2$ for $i = 1, 2$.*

We first catalogue the properties of these groups which we shall need. Almost all of the information which we need is contained in [18] and in the papers [5, 6] of Aschbacher and Seitz.

LEMMA 4.3. *Let K be a quasi-simple group with $K/Z(K) \cong L_3(4)$. Let $K \trianglelefteq \langle K, \alpha\rangle$ with $\alpha^2 = 1$, $\alpha \notin Z(\langle K, \alpha\rangle)$. The following properties are true:*

(1) *If α induces an inner automorphism of K, then $C_K(\alpha)$ has 2-rank at least 3 and if $C_K(\alpha)$ has 2-rank 3, then $O_2(K)$ has exponent 4.*

(2) *If α induces an outer automorphism of K, then one of the following holds:*

(a) *$C_K(\alpha)$ is non-solvable.*

(b) *$C_{\langle K,\alpha\rangle}(\alpha)$ has a Sylow 2-subgroup of exponent 4 and 2-rank 3.*

(c) *$C_K(\alpha)$ has non-abelian Sylow 2-subgroups.*

(d) *$m_2(C_{\langle K,\alpha\rangle}(\alpha)) = 4$.*

(3) *If $O_2(K) \cong \mathbb{Z}_4$ and $u \in I(Z(K))$, $u_1 \in I(K - \langle u\rangle)$, then $I(K) = \{u\} \cup u_1{}^K$.*

(4) *If α centralizes a Sylow 2-subgroup of K, then α induces an inner automorphism of K.*

(5) *If $O_2(K) = \langle 1\rangle$, then K has a 2-local subgroup N with $|N|_2 = |K|_2$ and $Z(N) = Z(K)$.*

Proof. By Proposition 2.3 of [18], if $O_2(K)$ is elementary, then every involution of K lies in an elementary subgroup of rank $4 + m(O_2(K))$. By Proposition 2.1 of [18], if $R \in \mathrm{Syl}_2(K)$, $Z = R \cap Z(K)$ and $\bar{X} = Z(R/Z)$, then the inverse image of $\bar{X}$ in R is abelian of rank $2 + m(Z)$. As every element of order 2 in R/Z is conjugate to an element of $\bar{X}$, (1) holds.

By Proposition 2.2 of [18], if α induces a graph or field automorphism of K, then $C_K(\alpha)$ is nonsolvable. If $O_2(K)$ has 2-rank 2 and α induces a unitary automorphism of K, then $m_2(C_{\langle K,\alpha\rangle}(\alpha)) = 4$. If $O_2(K)$ is cyclic, then $C_K(\alpha) \cong Q_8$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$. Thus (2) holds.

Suppose that $O_2(K) \cong \mathbb{Z}_4$. As $K/O_2(K)$ has one class of involutions, K has at most three classes of involutions with representatives u, u_1, uu_1. By Proposition 2.3 of [18], u_1 is not 2-central in K. Thus $u_1 \sim uu_1$ in K, proving (3).

By Proposition 2.2 of [18], if α induces an outer automorphism of K, then $|C_{K/Z(K)}(\alpha)|_2 \leqslant 8$, proving (4).

For (5), we may take $N = N_G(R)$ for $R \in \mathrm{Syl}_2(K)$, since by Burnside's Theorem, N is transitive on $Z(R)^{\#}$ and by Proposition 2.3 of [18], $C_K(R) = Z(R)\,Z(K)$.

LEMMA 4.4. *Let L be a quasi-simple group with $L/O(L) \cong Suz$. Let $L_0 = \mathrm{Aut}\, L$, $\bar{L} = \mathrm{Inn}\, L$.*

(1) $|L_0: \operatorname{Inn} L| = 2$.

(2) *L_0 has four classes of involutions*: $u^L \subseteq L$, $v^L \subseteq L$, $w_1{}^L$, $w_2{}^L$.

(3) *u is not 2-central.* $C_{\bar{L}}(\bar{u}) = (\bar{K} \times \bar{U})\langle \bar{x} \rangle$ *with* $\bar{K} \simeq L_3(4)$, $\bar{U} \simeq E_4$, $\langle \bar{U}, \bar{x} \rangle \simeq D_8$, *$\bar{x}$ inducing a unitary automorphism of $\bar{K}$.* $C_{\bar{L}}(\bar{K}) \simeq A_4 \simeq C_{\operatorname{Aut} L}(\bar{K})$.

(4) *v is 2-central.* $O_2(C_{\bar{L}}(\bar{v}))$ *is extra-special of order* 2^7. $C_{\bar{L}}(\bar{v})/O_2(C_{\bar{L}}(\bar{v})) \simeq \Omega^-(6, 2)$ *acts irreducibly on* $O_2(C_{\bar{L}}(\bar{v}))/\langle \bar{v} \rangle$.

(5) *w_1 and w_2 are outer and not 2-central.* $C_{\bar{L}}(w_1) \simeq \operatorname{Aut} HJ$; $C_{\bar{L}}(w_2) \simeq \operatorname{Aut} M_{12}$.

(6) *For all* $s \in I(L_0)$, $Z(C_{\langle \bar{L}, \bar{s} \rangle}(\bar{s})) = \langle \bar{s} \rangle$.

(7) $u^L \cap C_L(w_i) \neq \varnothing$ *for* $i \in \{1, 2\}$.

(8) *Let* $\bar{R} \in \operatorname{Syl}_2(\bar{K})$, $\bar{N} = N_{\bar{K}}(\bar{R})$. *Then* $C_{L_0}(\bar{N}) = C_{L_0}(\bar{K}) = A_4$.

Proof. Most of the facts in (1)–(5) are found in (16.5) of [6]. We need only add that $\Omega^-(6, 2)$ clearly must act irreducibly on $O_2(C_{\bar{L}}(\bar{v}))/\langle \bar{v} \rangle$. Fact 6 is immediate from (3)–(5). It is known that $C_L(w_i)$ has an involution u_i with $L(C_L(\langle u_i, w_i \rangle)) \simeq A_5$. By the L-Balance Theorem, $L(C_L(u_i)) \neq \langle 1 \rangle$. Thus $u_i \in u^L$. This proves (7). By 4.3(5), $C_{L_0}(\bar{N}) \cap N_{L_0}(\bar{K}) = C_{L_0}(\bar{K}) \simeq A_4$. Thus $C_{L_0}(\langle \bar{N}, \bar{u}_1 \rangle) = \bar{U}$ for each $\bar{u}_1 \in \bar{U}^{\#}$. Hence $C_{L_0}(\bar{N}) \simeq A_4$ or A_5. We suppose the latter and let $f \in C_{L_0}(\bar{N})$ of order 5. Let $\bar{v} \in Z(\bar{R})^{\#}$. Then $\bar{v}$ is a 2-central involution and $|C_{\bar{L}}(\langle \bar{v}, f \rangle)|_2 \geq 2^6$. As a subgroup of order 5 is self-centralizing in $O^-(4, 2) \simeq S_5$, $C_{\bar{L}}(\langle \bar{v}, f \rangle)$ is isomorphic to a subgroup of $\mathbb{Z}_5 \times GL(2, 3)$, a contradiction. This proves (8).

Definition 4.5. Let L be a quasi-simple group with $\bar{L} = L/O(L)$ of *O'Nan–Sims type*. Let $L_0 = \operatorname{Aut} L$.

(1) $|L_0: \operatorname{Inn} L| \leq 2$.

(2) L has one class of involutions, u^L, with $C_{\bar{L}}(u) = \bar{K}\langle \bar{x} \rangle$, $Z(\bar{K})$ cyclic of order 4, $\bar{K}/Z(\bar{K}) \simeq L_3(4)$, $\bar{x}$ of order 2 inverting $Z(\bar{K})$ and inducing a unitary automorphism on $\bar{K}/Z(\bar{K})$.

(3) If $L_0 \neq \operatorname{Inn} L$, then L_0-$\operatorname{Inn} L$ has one class of involutions, w^L, with $C_{\bar{L}}(w) \simeq J_1$.

(4) $|\bar{L}| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$.

Remark. These properties are found in [25].

Theorem 4.6. *Let G be a finite group with $O(G) = \langle 1 \rangle$ having a standard subgroup K with $K/Z(K) \simeq L_3(2^n)$, $n \geq 2$, Suz or He. If $K/Z(K) \simeq$ Suz, assume that $|Z(K)|$ is even. Then one of the following holds*:

(1) $K \subseteq K_1K_2 \trianglelefteq G$ *with* $K_i \trianglelefteq K_1K_2$, $K_i \simeq K$.

(2) $K \simeq L_3(2^n)$, $\langle K^G \rangle \simeq L_3(2^{2n})$ *and* $C_G(K) \simeq \mathbb{Z}_2$.

(3) $K \simeq L_3(4)$ *and* $\langle K^G \rangle \simeq Suz$.

(4) $Z(K) \simeq E_4$, $K/Z(K) \simeq L_3(4)$ *and* $\langle K^G \rangle \simeq He$.

(5) $Z(K) \simeq Z_4$, $K/Z(K) \simeq L_3(4)$ *and* $\langle K^G \rangle$ *is of O'Nan–Sims type.*

In particular, if $F^*(G)$ *is simple, there is no* $t \in I(\mathrm{Aut}\, G)$ *and* $L \trianglelefteq C_G(t)$ *with* $L/Z(L) \simeq Suz$ *or* $L/Z(L)$ *of O'Nan–Sims type.*

Proof. These results are due to Nah [24], Seitz [28], O'Nan [25], Griess and Solomon [18] and Finkelstein and Solomon [11].

DEFINITION 4.7. Let L be a simple group of F_5 *type*, $L_0 = \mathrm{Aut}\, L$.

(1) $|\mathrm{Aut}\, L \colon \mathrm{Inn}\, L| \leqslant 2$.

(2) L has two classes of involutions: u^L and v^L.

(3) u is not 2-central. $C_L(u) = K\langle x \rangle$, where K is a perfect 2-fold covering group of HiS, $x^2 = 1$ and $K\langle x \rangle/Z(K) \simeq \mathrm{Aut}\, HiS$. $C_L(K) = \langle u \rangle$. If $L \neq \mathrm{Aut}\, L$, then $C_{\mathrm{Aut}\, L}(K)$ is cyclic of order 4 and is inverted by x.

(4) v is 2-central. $O_2(C_L(v))$ is extra-special of order 2^9. $C_L(v)/O_2(C_L(v)) \simeq A_5 \wr \mathbb{Z}_2$, acting irreducibly of $O_2(C_L(v))/\langle v \rangle$. $Z(C_{\mathrm{Aut}\, L}(v)) = \langle v \rangle$.

(5) If $\mathrm{Aut}\, L \neq \mathrm{Inn}\, L$, then $\mathrm{Aut}\, L - \mathrm{Inn}\, L$ has one class of involutions, w^L, with $C_L(w) \simeq S_{10}$.

(6) $u^L \cap C_L(w) \neq \varnothing$.

(7) $|L| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$.

Remark. The properties listed above may all be found in [20]. In particular, if $\mathrm{Aut}\, L \neq \mathrm{Inn}\, L$, then $\mathrm{Aut}\, L \supseteq S_{12}$ and if w_1, w_2 are commuting transpositions in S_{12}, then $w_i \in w^L$ and $w_1 w_2 \in u^L \cap C_L(w_i)$.

LEMMA 4.8. *Let* $K = HiS$, $\bar{K} = K/Z(K)$, $K_0 = \mathrm{Aut}\, K$.

(1) $\bar{K}$ *has two classes of involutions* $u_1{}^L$ *and* $v_1{}^L$. $C_{\bar{K}}(u_1) \simeq \mathbb{Z}_2 \times \mathrm{Aut}\, A_6$; $C_{\bar{K}}(v_1)/O_2(C_{\bar{K}}(v_1)) \simeq S_5$ *and* $|O_2(C_{\bar{K}}(v_1))| = 2^6$.

(2) $\mathrm{Aut}\, K - \mathrm{Inn}\, K$ *has two classes of involutions* $w_1{}^L$ *and* $w_2{}^L$. $C_{\bar{K}}(w_1) \simeq S_8$. $C_{\bar{K}}(w_2)/O_2(C_{\bar{K}}(w_2)) \simeq S_5$ *and* $O_2(C_{\bar{K}}(w_2)) \simeq E_{16}$. $|C_{\bar{K}}(w_i)| = 2^7 < 2^9 = |\bar{K}|_2$.

(3) $m_2(K) \geqslant 3$.

Proof. Facts 1 and 2 may be found in Section 12 of [6]. As u_1 is not a square in $\bar{K}$, every involution of $E(C_{\bar{K}}(u_1))$ is 2-central. $\bar{K}$ has a transitive representation of degree 100 and v_1 fixes exactly 8 points. Thus v_1 lifts to an involution in $K \subseteq \hat{A}_{100}$. Thus $E(C_{\bar{K}}(u_1))$ lifts to $\mathbb{Z}_2 \times A_6$, proving (3).

THEOREM 4.9. *Let* G *be a finite group with* $O(G) = \langle 1 \rangle$ *having a standard subgroup* K *with* $K/Z(K) \simeq HiS$. *Then one of the following holds*:

(1) $K \subseteq K_1K_2 \trianglelefteq G$ *with* $K_i \trianglelefteq K_1K_2$, $K_i \cong K$.

(2) $K \cong \hat{H}iS$ *and* $\langle K^G \rangle$ *is of type* F_5.

Proof. This is a consequence of the main theorems of [20, 30].

LEMMA 4.10. *Let* $K = G_2(4)$, $K_0 = \operatorname{Aut} K$. *Let* $\hat{K}$ *be a perfect* 2-*fold covering group,* $\hat{G}_2(4)$, *of* $G_2(4)$.

(1) $|K_0 : \operatorname{Inn} K| = 2$.

(2) *K has two classes of involutions a^K and b^K, where a is a long root involution, b a short root involution.*

(3) $C_K(a)/O_2(C_K(a)) \cong SL(2,4) \cong C_K(b)/O_2(C_K(b))$. *If* $R \in \mathrm{Syl}_2(K)$, *then* $Z(R)^{\#} \subseteq a^K$. *If* $a \in Z(R)$, *then* $Z(R) = Z(C_K(a)) \cong E_4$. $C_K(a) = C_K(a)'$. $O_2(C_K(b))$ *is elementary. Neither* $C_K(a)$ *nor* $C_K(b)$ *has a normal subgroup isomorphic to* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_4$.

(4) $K_0 - \operatorname{Inn} K$ *has one class of involutions,* c^{K_0}, *with* $C_K(c) \cong G_2(2) \cong \operatorname{Aut} U_3(3)$.

(5) $|G_2(2)| = 2^6 \cdot 3^3 \cdot 7$. $|G_2(4)| = 2^{12} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13$.

(6) *The subgroup H of K generated by the long root subgroups is isomorphic to $SL(3,4)$. If $r \in Z(H)$ of order 3, then $H = C_K(r)$. Also $r^K \cap C_K(c) \neq \emptyset$. The inverse image $\hat{H}$ of H is $\hat{K}$ is a perfect 6-fold covering group of $PSL(3,4)$.*

(7) *b lifts to an element of order 4 in $\hat{K}$.*

(8) *If $r_1 \in K$ of order 3, then $C_K(r_1)$ is nonsolvable.*

Proof. Facts (1) and (4) are in (19.2) of [5]. Fact (2) and most of (3) are in (18.4) of [5]. The fact that $Z(R) = Z(C_K(a)) \cong E_4$ may be deduced from the defining commutator relations, given in (3.1) of [31]. As $O_2(C_K(b))$ is elementary, $C_K(b)$ has no normal subgroup isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$. Suppose $T \trianglelefteq C_K(a)$ with $T \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$. As $Z(C_K(a)) \cong E_4$, $C_K(T) \trianglelefteq C_K(a)$ with $|C_K(a) : C_K(T)| = 2^n > 1$. But $C_K(a) = C_K(a)'$, a contradiction. This proves (3). Fact 5 is well known. Generators are relations for $\hat{K}$ are given in Chapter I of [15]. From these, (7) is immediate and it is easily checked that the inverse images in $\hat{K}$ of the long root subgroups generate a perfect 6-fold covering group of $PSL(3,4)$. In particular, as $f_{r,s}(1,\omega) \neq 1$ for $\omega \in GF(4) - \{0,1\}$ and r, s long roots inclined at 120° to each other, the extension of $\mathbb{Z}_2$ by $SL(3,4)$ is nonsplit in $\hat{K}$. From [7], we see that $C_K(r) = H$. As $|G_2(2)|_3 = |G_2(4)|_3$, $r^K \cap C_K(c) \neq \emptyset$. Thus Fact (6) holds. Fact (8) may be found in [7].

LEMMA 4.11. *Let* $L = \cdot 1$.

(1) $L = \operatorname{Aut} L$.

(2) *L has three classes of involutions u^L, v^L, w^L.*

(3) *$C_L(u) = (K \times U)\langle x\rangle$ where $K \cong G_2(4)$, $U \cong E_4$, $x^2 = 1$, $K\langle x\rangle \cong$ Aut $G_2(4)$ and $U\langle x\rangle \cong D_8$. Also $C_L(K) \cong A_4$. u is not 2-central.*

(4) *v is 2-central. $O_2(C_L(v))$ is extra-special of order 2^9. $C_L(v)/O_2(C_L(v)) \cong \Omega^+(8, 2)$ acting irreducibly on $O_2(C_L(v))/\langle v\rangle$.*

(5) *w is not 2-central. $O_2(C_L(w)) \cong E_{2^{11}}$. $C_L(w)/O_2(C_L(w)) \cong$ Aut M_{12} acting indecomposably on $O_2(C_L(w))$ and irreducibly on $O_2(C_L(w))/\langle w\rangle$.*

(6) *Let $R \in \mathrm{Syl}_2(K)$, $N = N_K(R)$. Then $C_L(N) = C_L(K) \cong A_4$.*

Proof. Most of the information in (1)–(5) is in (16.8) of [6]. As the involutions of U are L-conjugate and $K\langle x\rangle \cong \mathrm{Aut}\, K$, it follows that $C_L(K) \cong A_4$. If $E = O_2(C_L(w))$, then $N_G(E)$ is a splitting extension of E by M_{24}, the so-called Conway module. Fact 5 now follows from well-known properties of this module.

Finally, we consider (6). By Lemma 4.10 and Burnside's lemma, $C_K(N) = \langle 1\rangle$. Thus $C_L(N) \cap C_L(u) = U$ for each $u \in U^{\#}$ and $C_L(N) \cap N_L(U) \cong A_4$. Thus either $C_L(N) \cong A_4$ or A_5. Suppose that $r \in C_L(N)$ of order 5. We may assume that $r \in C_L(v)$ for some 2-central involution v of L with $2^{12} \mid |C_L(\langle r, v\rangle)|$. Then clearly r has exactly one Jordan block and $C_L(\langle r, v\rangle)/O_2(C_L(\langle r, v\rangle))$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_5 \times GL(4, 2)$. Also $|O_2(C_L(\langle r, v\rangle))| = 2^5$. But then $|C_L(\langle r, v\rangle)|_2 = 2^{11}$, a contradiction. Hence $C_L(N) \cong A_4$, proving (6).

THEOREM 4.12. *Let G be a finite group with $O(G) = \langle 1\rangle$ having a standard subgroup K with $K/Z(K) \cong G_2(2^n)$ for some $n \geq 2$. Then one of the following holds:*

(1) *$K \subseteq K_1K_2 \trianglelefteq G$ with $K_i \trianglelefteq K_1K_2$, $K_i \cong K$.*

(2) *$K \cong G_2(2^n)$, $\langle K^G\rangle \cong G_2(2^{2n})$ and $C_G(K) \cong \mathbb{Z}_2$.*

(3) *$K \cong G_2(4)$ and $\langle K^G\rangle \cong .1$.*

Proof. By [6, 34], we may assume that $C_G(K)$ has cyclic Sylow 2-subgroups and by Yamada [32], we may assume that $n = 2 = |Z(K)|$. By 4.10(3) and Theorem 2.4(2), $Z(K) \in \mathrm{Syl}_2(C_G(K))$. Let $Z(K) = \langle t\rangle$. By Glauberman's Z^*-Theorem, $t^G \cap C_G(t) \neq \{t\}$. Using the notation of Lemma 4.10, we have $g \in G$ with either $t^g \in a^K$ or $t^g \in c^K$. In particular, by 4.10(6), there exists $r \in C_K(t^g)$ of order 3 with $C_K(r)$ a perfect 6-fold covering group of $L_3(4)$. It follows that $C_K(r)$ is a standard subgroup of $C_G(r)$. Then by Theorem 4.6, $t \in Z^*(C_G(r))$. Thus $C_G(\langle r, t^g\rangle)$ is 2-nilpotent, since t^g induces a unitary automorphism on $C_K(r)$. But $C_{K^g}(r)$ is nonsolvable by 4.10(8), a contradiction. This proves Theorem 4.12.

DEFINITION 4.13. Let L be a simple group of *type* F_2.

(1) $L = \mathrm{Aut}\, L$. The Schur multiplier of L has order at most 2. If $\hat{L}/Z(\hat{L}) \cong L$, then $m_2(\hat{L}) \geq 3$.

(2) L has four classes of involutions: u^L, v^L, $w_1{}^L$, $w_2{}^L$.

(3) u is not 2-central. $C_L(u) = K\langle x\rangle$ with K a perfect 2-fold covering group of ${}^2E_6(2)$ and $K\langle x\rangle/\langle u\rangle \simeq \mathrm{Aut}\,{}^2E_6(2)$.

(4) v is 2-central. $O_2(C_L(v))$ is extra-special of order 2^{23} and $C_L(v)/O_2(C_L(v)) \simeq .2$ acting irreducibly on $O_2(C_L(v))/\langle v\rangle$.

(5) w_1 is not 2-central. $C_L(w) = (K_1 \times W_1)\langle x_1\rangle$ with $K \simeq F_4(2)$, $W_1 \simeq E_4$, $x_1^2 = 1$ and $\langle W_1, x_1\rangle \simeq D_8$. $W_1 - \langle w_1\rangle \subseteq u^L$.

(6) w_2 is not 2-central. $C_L(w_2)/O_2(C_L(w_2)) \simeq O_8^+(2)$. $C_L(w_2) \cap u^L \neq \varnothing$ and $\langle w_2\rangle = Z(C_L(w_2))$.

(7) $|L| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$.

Remark. Most of these facts are explicitly listed in (15.3) of [6]. In particular, by 15.3(3), we may pick $w_2 \in C_L(u)$. If $\hat{L}/Z(\hat{L}) \simeq L$, then $\hat{L}$ contains a copy of .2, $m_2(\hat{L}) \geqslant 3$.

DEFINITION 4.14. Let L be a simple group of *type* F_1.

(1) $L = \mathrm{Aut}\,L$.

(2) L has two classes of involutions: u^L, v^L.

(3) u is not 2-central. $C_L(u)$ is a perfect 2-fold covering group of a group of type F_2.

(4) $O_2(C_L(v))$ is extra-special of order 2^{25} and $C_L(v)/O_2(C_L(v)) \simeq .1$, acting irreducibly on $O_2(C_L(v))/\langle v\rangle$.

(5) $|L| = 2^{46} \cdot 3^{22} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$.

LEMMA 4.15. *Let* $K = {}^2E_6(2)$, $\hat{K} = {}^2\hat{E}_6(2)$, $K_0 = \mathrm{Aut}\,K$.

(1) $|K_0 : \mathrm{Inn}\,K| = 2$ *and* $m_2(\hat{K}) \geqslant 3$.

(2) *K has three classes of involutions*: a^K, b^K, c^K.

(3) *$C_K(a)$ involves* $SU(6, 2)$. *$C_K(b)$ involves* $\mathrm{Sp}(6, 2)$ *and* $O_2(C_K(c))$ *faithfully admits* $S_3 \times S_3$.

(4) $K_0 - \mathrm{Inn}\,K$ *has two classes of involutions*: d^{K_0}, e^{K_0}.

(5) $C_K(d) \simeq F_4(2)$ *and* $C_K(e) \simeq C_{F_4(2)}(t)$ *for t a 2-central involution of* $F_4(2)$. *In particular, $C_K(e)$ involves* $\mathrm{Sp}(6, 2)$. *Neither d nor e is 2-central in* $\hat{K}\langle d\rangle$.

Proof. As $\hat{K}$ contains a copy of $SU(6, 2)$, $m_2(\hat{K}) \geqslant 3$. Facts (2)–(5) may be deduced from (13.1), (14.1), (14.3) and (19.9) of [5]. In particular as $|F_4(2)|_2 = 2^{24}$ and $|{}^2E_6(2)|_2 = 2^{36}$, neither d nor e is 2-central in $\hat{K}\langle d\rangle$.

THEOREM 4.16. *Let G be a finite group with $F^*(G)$ quasi-simple having a subgroup K with $Z(K) \subseteq Z(G)$, $|Z(K)| = 2$ and $KZ(G)/Z(G)$ standard in $G/Z(G)$. If $K/Z(K) \simeq {}^2E_6(2)$, then $K = F^*(G)$.*

Proof. By Seitz [29], $F^*(G)/Z(G) \cong {}^2E_6(2)$ or $E_6(4)$. Since the Schur multiplier of $E_6(4)$ has order three, the former case holds.

Remark. Using Theorem 4.15, we see that if G is a minimal counterexample to Theorem 4.2 and H is a quasi-simple section of G with $K \subseteq H$, $|Z(K)| = 2$, $K/Z(K) \cong {}^2E_6(2)$ or $K/Z(K)$ of F_2 type and $KZ(H)/Z(H)$ standard in $H/Z(H)$, then either $H/Z(H) \cong {}^2E_6(2)$ or $H/Z(H)$ is of F_2 or F_1 type.

With this information in hand we are ready to begin checking the conditions for Hypotheses 3.1, 3.5 and 3.8. We apologize in advance for certain "abuse of notation" in the use of the letters K and L. We begin by fixing G as a minimal counterexample to either Theorem 4.1 or Theorem 4.2. We let L be a standard subgroup of G of "type L" with $Z(L) = O(L)$. We let K be a standard subgroup of L conforming to the notation of Lemmas 4.4 and 4.11 and Definitions 4.5, 4.7, 4.13 and 4.14.

LEMMA 4.17. *Hypothesis* 3.1(1) *holds.* $|G : F^*(G)| \leqslant 2$.

Proof. A standard argument using the L-Balance Theorem shows that $F^*(G)$ is simple $|G : F^*(G)| \leqslant 2$. By Lemmas 4.3(1), 4.8(3) and 4.15(1) and Definition 4.13(1), $m_2(K) \geqslant 3$ or $K \cong G_2(4)$. By Lemma 4.10(6), $SL(3, 4) \subseteq G_2(4)$. Thus by Lemma 4.3(1), $m_2(G_2(4)) \geqslant 3$.

LEMMA 4.18. *Hypothesis* 3.1(2) *holds. If* $L_1 \cong .1$, *then* $|C_G(L_1)|_2 = 2$.

Proof. By minimality of G, if $L_1 \in \mathscr{L}(G)$ with L_1 of type L, then $L_1 \in \mathscr{L}^*(G)$. As $m_2(L_1) \geqslant 3$, it follows by Theorem 2.2 that L_1 is standard in G. Now by Theorem 2.4(1,3), $C_G(L_1)$ has cyclic Sylow 2-subgroups. By Lemma 4.11, if $L_1 \cong .1$ and $s \in I(\operatorname{Aut} L_1)$, then $O(C_G(s)) = \langle 1 \rangle$ and any abelian normal subgroup of $C_G(s)$ is elementary. Thus by Theorem 2.4(2), $|C_G(L)|_2 = 2$.

LEMMA 4.19. *Hypothesis* 3.1(3) *holds.*

Proof. The argument is uniform in all cases, but is perhaps more transparent when illustrated in a particular case. We shall argue in the most complicated case, i.e., when $K/Z(K) \cong L_3(4)$.

We have $K_1 \in \mathscr{L}(G)$, $Z_1 \subseteq Z(K_1)$ with $K_1/Z_1 \cong K$. Suppose that $F^*(G) \subseteq G_1 \subseteq G$ with $K_1 \in \mathscr{L}^*(G_1)$. Then G_1 is known by Theorem 4.6, and in particular, there is no $L \in \mathscr{L}(G)$ with $L/Z(L) \cong Suz$ or $L/Z(L)$ of O'Nan–Sims type. This contradicts our assumptions. Thus $K_1 \notin \mathscr{L}^*(G_1)$.

Suppose that $K_1 \lll L_1 \in \mathscr{L}(G)$ with $L_1/Z(L_1) \cong K_1/Z(K_1)$. We may apply Corollary 2.3 with L_1 in place of G, K_1 in place of K to deduce that there exists a chain

$$K_1, K_2, \ldots, K_{n-1}, K_n = L_1$$

satisfying Conditions (1)–(5) of Corollary 2.3. As $L_1(Z(L_1) \cong K_1/Z(K_1)$ there exists $i \geqslant 2$ with $K_{i-1}/Z(K_{i-1}) \cong L_3(4)$ and s_i a 2-element of $N_G(K_i)$ such that

$K_{i-1}Z(K_i)/Z(K_i)$ is standard in $\langle K_i, s_i\rangle/Z(K_i)$. By Theorem 4.6, $K_i/Z(K_i)$ is isomorphic to $L_3(16)$, *Suz*, *He* or $K_i/Z(K_i)$ is of O'Nan–Sims type. Repeating this argument and using the minimality of G, we conclude that $L_1/Z(L_1)$ is isomorphic to $L_3(2^n)$ for some $n \geqslant 4$, *Suz* or *He* or $L_1/Z(L_1)$ is of O'Nan–Sims type. We now apply Corollary 2.3 with L_1 in place of K_1 to deduce that there exists a chain

$$L_1, L_2, \dots, L_{n-1}, L_n = F^*(G)$$

satisfying Conditions (1)–(5) of Corollary 2.3. As $F^*(G)$ is simple, L_{n-1} is standard in G. Thus by Theorem 4.6, L_{n-1} is not isomorphic to $L_3(2^n)$ for any $n \geqslant 4$ and $L_{n-1} \not\cong He$. Moreover if $L_{n-1}/Z(L_{n-1}) \cong Suz$, then $|Z(L_{n-1})|$ is odd. We conclude that $|Z(L_{n-1})|$ is odd and either $L_{n-1}/Z(L_{n-1}) \cong Suz$ or $L_{n-1}/Z(L_{n-1})$ is of O'Nan–Sims type. As L_{n-1} is standard in G, $[L_{n-1}, (L_{n-1})^g] \neq \langle 1\rangle$ for all $g \in G$. Thus Condition (5b) of Corollary 2.3 must hold when $i = n-1$, if $n - 1 > 1$. But them $L_{n-2}/Z(L_{n-2}) \cong L_3(4)$, contrary to the fact that $|L_1/Z(L_1)| > |L_3(4)|$. Thus $n = 2$, i.e., $L_1 \in \mathscr{L}^*(G)$ and $L_1/Z(L_1) \cong Suz$ with $|Z(L_1)|$ odd or $L_1/Z(L_1)$ is of O'Nan–Sims type. This verifies Hypothesis 3.1(3) in the case $K/Z(K) \cong L_3(4)$. Clearly the other cases follow in the same way.

LEMMA 4.20. *Hypotheses* 3.1 *holds. Also if* $L/O(L) \cong Suz$, *then* $|C_G(L)|_2 = 2$.

Proof. It remains to verify Condition 3.1(4). Condition (4b) is immediate from Lemmas 4.4(3) and 4.11(3) and Definitions 4.5(2), 4.7(3), 4.13(3) and 4.14(3). Indeed we list below $C_{\mathrm{Aut}\,L}(K)$ for each case:

L	K	$C_{\mathrm{Aut}\,L}(K)$
Suz	$L_3(4)$	$A_4 \subseteq L$
$O'NS$	$\mathbb{Z}_4 \cdot L_3(4)$	$\mathbb{Z}_4 \subseteq L$
F_5	$\mathbb{Z}_2 \cdot HiS$	$\mathbb{Z}_4$
A	$G_2(4)$	$A_4 \subseteq L$
F_2	$2 \cdot {}^2E_6(2)$	$\mathbb{Z}_2 \subseteq L$
F_1	$2 \cdot F_2$	$\mathbb{Z}_2 \subseteq L$

Now $C_G(L) \trianglelefteq N_G(L) \cap C_G(K)$ and $C_G(L)$ has cyclic Sylow 2-subgroups. Thus $C_G(L)$ is solvable. As $N_G(L) \cap C_G(K)/C_G(L)$ is isomorphic to a subgroup of $C_{\mathrm{Aut}\,L}(K)$, $N_G(L) \cap C_G(K)$ is solvable of 2-rank $1 + m_2(C_L(K)) \leqslant 3$. By Lemmas 4.8 and 4.10 and Definition 4.13, if α is an involutory automorphism of a central extension K_2 of K, then either $C_{K_2}(\alpha)$ is nonsolvable or $K_2/Z(K_2) \cong L_3(4)$ or ${}^2E_6(2)$. Moreover, by Lemma 4.15, if $K_2/Z(K_2) \cong {}^2E_6(2)$, then $O_2(C_{K_2}(\alpha))$ faithfully admits $S_3 \times S_3$. A Sylow 2-subgroup of $N_G(L) \cap C_G(K) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ for some $n \geqslant 2$, if L is of type F_2. No subgroup of this 2-group faithfully admits $S_3 \times S_3$. Thus if Condition (4c) does not hold, then $K/Z(K) \cong L_3(4)$. By

Lemma 4.3(1,2), if $K_2/Z(K_2) \cong L_3(4)$ and α induces an involutory automorphism of K_2 with $C_{\langle K_2,\alpha\rangle}(\alpha)$ solvable of 2-rank $\leqslant 3$, then one of the following holds:

(1) $C_{K_2}(\alpha)$ has non-abelian Sylow 2-subgroups.

(2) $m_2(C_{\langle K_2\alpha\rangle}(\alpha)) = 3$ and $C_{K_2}(\alpha)$ has Sylow 2-subgroups of exponent 4.

As $N_G(L) \cap C_G(K) \subseteq LC_G(L)$, $N_G(L) \cap C_G(K)$ has abelian Sylow 2-subgroups.

It remains for us to check (4a). If $|Z(K)|$ is even, we may take $N = K$. Then (4ai) is trivial, since $N_G(L) \cap C_G(K)$ is solvable. If $L/O(L) \cong$ *Suz* or .1, we let $R \in \mathrm{Syl}_2(K)$, $N = N_K(R)$. Then by Lemmas 4.4(8) and 4.11(6), $C_{\mathrm{Aut}\,L}(N) = C_{\mathrm{Aut}\,L}(K)$. Again, as $N_G(L) \cap C_G(K)$ has abelian Sylow 2-subgroups, (4ai) is trivial. Moreover, if Condition (4a) fails, then Condition (2) of the previous paragraph holds and $m_2(N_G(L) \cap C_G(K)) \geqslant 3$. Thus $L/O(L) \cong$ *Suz* and a Sylow 2-subgroup of $N_G(L) \cap C_G(K)$ is $U \times \langle t_0\rangle$ where $U \cong E_4$ and $\langle t_0\rangle \in \mathrm{Syl}_2(C_G(L))$ As $C_{K_2}(\alpha)$ has Sylow 2-subgroup of exponent 4, we shall be done once we show that $\langle t_0\rangle$ has order 2. By Lemma 4.4, if $t_1 \in I(\mathrm{Aut}\,L)$, then $O(C_L(t_1)) = \langle 1\rangle$ and a normal abelian 2-subgroup of $C_{\mathrm{Aut}\,L}(t_1)$ is isomorphic to $\mathbb{Z}_2$ or E_4. Thus, by Theorem 2.4(a), t_0 has order 2 and we are done.

This completes the proof of Hypothesis 3.1.

As Hypothesis 3.1 holds, we can find a K-extremal pair (K, L) satisfying Lemma 3.3 with $|C_G(L)|_2$ maximal. We adopt the notation of Notation 3.4.

LEMMA 4.21. *Hypothesis* 3.5 *holds.*

Proof. By Lemma 3.6, this is true if $u \in K$. Thus we may assume that $L/O(L) \cong$ *Suz* or .1. By Lemmas 4.4(1) and 4.11(1), $|\mathrm{Aut}\,L{:}\,\mathrm{Inn}\,L| \leqslant 2$. By Lemmas 4.18 and 4.20, $|C_G(L)|_2 = 2$. By Lemmas 4.4(3) and 4.11(3), $u \in C_G(\langle u, t\rangle)'$. Thus by Lemma 3.6, Hypothesis 3.5 holds.

LEMMA 4.22. *Condition* (1) *of Lemma* 3.9 *holds.*

Proof. Let $S \in \mathrm{Syl}_2(C_L(u))$, $R = S \cap K$. By Lemmas 4.3(4), 4.8(2), 4.10(4,5) and 4.15(5) and Definition 4.13(1), $Z(S) \subseteq RC_S(K)$. By Lemmas 4.4(3) and 4.11(3) and Definitions 4.7(3), 4.13(3) and 4.14(3), either $S \notin \mathrm{Syl}_2(L)$ or $L/O(L)$ is of O'Nan–Sims type. If L is of type F_5, F_2 or F_1, then by Definitions 4.7(3), 4.13(3) and 4.14(3), $Z(S) \subseteq R$ and $Z(K) = \langle u\rangle$. Thus 3.10(3) holds. If $L/O(L) \cong$ *Suz* or .1, then by Lemmas 4.4(3) and 4.11(3), $Z(S) \subseteq R \times C_S(K)$ and $C_G(K)$ is transitive on $C_S(K)^{\#}$. Thus 3.10(4) holds. Finally if $L/O(L)$ is of O'Nan–Sims type, then by Definition 4.5(2), $I(R) \subseteq u^L$. Thus 3.10(1) holds. Now by Lemma 3.10, Condition 3.9(1) holds.

LEMMA 4.23. *Condition* (2) *of Lemma* 3.9 *holds.*

Proof. By Lemma 4.11(3) and Definitions 4.7(3), 4.13(3) and 4.14(3), $C_L(u)/C_L(K) \cong \mathrm{Aut}\,K$ unless $K/Z(K) \cong L_3(4)$. If $K/Z(K) \cong L_3(4)$ and

$Z(K) = O(K)$, then K has one class of involutions and if $O_2(K) \cong \mathbb{Z}_4$, then $K - Z(K)$ has one class of involutions by Lemma 4.3(3). This proves 3.9(2).

LEMMA 4.24. *Hypothesis* 3.8 *holds.*

Proof. By Lemmas 3.9, 3.11, 4.22 and 4.23, it suffices to verify the two conditions of Lemma 3.11. By Lemmas 4.4 and 4.11 and Definitions 4.5, 4.7, 4.13 and 4.14, if $K \cong K_1 \in \mathscr{L}(L)$, then $K_1 \in \mathscr{L}^*(L)$. Thus 3.11(1) holds. By Theorems 4.6, 4.9 and 4.12 and the hypotheses of Theorem 4.2, if L_1 is a simple group and K_1 is a standard subgroup of $\operatorname{Aut} L_1$ with $K_1 \cong K$ and $m_2(C_{\operatorname{Aut} L_1}(K_1)) \geqslant m_2(C_L(K))$, then $|L_1| = |L|$. Thus if L_1 is isomorphic to a section of L, then $L_1 \cong L$. Thus 3.11(2) holds.

LEMMA 4.25. *Condition* (1) *of Lemma* 3.12 *holds.*

Proof. By Lemmas 4.4(1) and 4.11(1) and Definitions 4.5(1), 4.7(1), 4.13(1) and 4.14(1), $|\operatorname{Aut} L : L| \leqslant 2$. Also by Lemmas 4.4 and 4.11 and Definitions 4.5, 4.7, 4.13 and 4.14, $\langle s \rangle = Z(C_{\operatorname{Aut} L}(s))$ for all $s \in I(\operatorname{Aut} L)$. Thus Condition (1) holds.

LEMMA 4.26. $L \cong .1$.

Proof. Suppose that $L \ncong .1$. We shall show that Condition (3) of Lemma 3.12 holds. Indeed, this is immediate from Lemma 3.13, Lemma 4.4(2,7) and Definitions 4.5(2), 4.7(2,5,6), 4.13(2–6) and 4.14(2). Thus Lemma 3.12 yields a contradiction.

LEMMA 4.27. $L \ncong .1$.

Proof. Suppose that $L \cong .1$. We shall show that Condition (2) of Lemma 3.12 holds. By Lemma 4.11(1), $L = \operatorname{Aut} L$. By Lemma 4.11(3–5), $s \in C_L(s)'$ for all $s \in I(L)$. Finally, by Lemma 4.11(3), $I(C_L(K)) \subseteq u^L$. Thus Lemma 3.12 yields a contradiction.

As Lemmas 4.26 and 4.27 are mutually contradictory, we have proved Theorems 4.1 and 4.2. In particular, we have proved Theorem 1.4.

5. THE RUDVALIS CASE

In this section we complete the proof of Theorem 1.1 by handling the case where $L \cong Ru$, the Rudvalis simple group. It will be convenient to use the full strength of the Unbalanced Group Theorem in this section. Thus G will be a finite balanced group which is a minimal counterexample to Theorem 1.1. We collect the needed properties of Ru and $Sz(8)$.

LEMMA 5.1. *Let* $K = Sz(8)$, $K_0 = \mathrm{Aut}\, K$, $R \in \mathrm{Syl}_2(K)$. *Let* $\hat{K}$ *denote any perfect covering group of* K.

(1) *R is a special 2-group of order 2^6 with $\Omega_1(R) = Z(R) \cong E_8$.*

(2) $K_0 = K\langle\rho\rangle$ *with* $\rho^3 = 1$. $C_K(\rho) = Sz(2)$, *a Frobenius group of order* 20.

(3) $N_{K_0}(R)/R$ *is a Frobenius group of order* 21 *transitively permuting* $Z(R)^{\#}$.

(4) *If* $\alpha \in \hat{K}$ *of order* 2, *then* $m_2(C_{\hat{K}}(\alpha)) \geqslant 3$ *and if* $m_2(C_{\hat{K}}(\alpha)) = 3$, *then* $\hat{K} = K$ *and* $C_K(\alpha) = R$.

Proof. See [1] for (1)–(3) and the fact that if $\hat{K}$ is the full covering group of K, then $Z(\hat{K}) \cong E_4$. Let $x \in N_K(R)$ of order 7 and suppose that the inverse image of $\Omega_1(R)$ in $\hat{K}$ is not elementary. Then every element of $\Omega_1(\hat{R}) - Z(\hat{K})$ has the same square $z_1 \in Z(\hat{K})^{\#}$. Let $z_2 \in Z(\hat{K}) - \langle z_1\rangle$. Then $\hat{R}/\langle z_2\rangle$ has order 16, exponent 4 and only one involution. This is impossible. Thus (4) holds.

LEMMA 5.2. *Let* $L = Ru$.

(1) $L = \mathrm{Aut}\, L$.

(2) *L has two classes of involutions u^L and v^L.*

(3) *u is not 2-central.* $C_L(u) = U \times K$, *where* $U \cong E_4$ *and* $K \cong Sz(8)$. $N_L(U) = N_L(K) = (U \times K)\langle r\rangle$ *where* $r^3 = 1$, $\langle U, r\rangle \cong A_4$ *and* $\langle K, r\rangle \cong \mathrm{Aut}\, Sz(8)$.

(4) *v is 2-central.* $C_L(v)/O_2(C_L(v)) \cong S_5$. $v \in C_L(v)'$ *and* $\langle v\rangle = Z(C_L(v))$. *Any normal abelian subgroup of $C_L(v)$ is elementary.*

(5) *Let r be as in* (3). *Then* $r \in C_L(r)'$ *and* $C_L(r)/\langle r\rangle \cong M_{10}$.

(6) *L has two classes of elements of order* 5: h^L *and* f^L. $C_L(h) = RQ$ *where R is an extra-special 5-group of order 5^3,* $Q \cong Q_8$ *and* $C_L(R) = Z(E)$. $C_L(f) \cong \mathbb{Z}_5 \times A_5$.

Proof. See [26]. Note that $I(K) \subseteq v^L$, so that if $v \in C_L(u)$, then $v \in C_L(\langle u, v\rangle)'$.
We also need the following elementary consequences of [13] and [19].

LEMMA 5.3. (1) *Let G be a finite group with* $O(G) = \langle 1\rangle$. *Assume that* $t \in I(G - Z(G))$ *and* $C_G(t) \cong \mathbb{Z}_2 \times M_{10}$. *Then either* $F^*(G) \cong U_4(3)$ *or* $F^*(G) \cong A_6 \times A_6$.

(2) *Let G be a finite group having $t \in I(G)$ with $C_G(t)/O_5(C_G(t))$ isomorphic to a subgroup of* $\mathbb{Z}_2 \times Q_8$. *Then G does not involve* $U_4(3)$.

Proof. In both cases, $C_G(t)$ contains a 2-group of order 8 which is self-centralizing in a Sylow 2-subgroup of G. Thus by [19], G has sectional 2-rank at most 4. In case 2, we may assume that $O(G) = \langle 1\rangle$. Suppose that K_1 is a component of $E(G)$ with $K_1{}^t \neq K_1$. Then Case 1 holds and $K_1/Z(K_1) \cong A_6$. As $t \notin F^*(G)$ in this case, $C_{F^*(G)}(t) \cong A_6$ or M_{10}. Thus $F(G) = \langle 1\rangle$ and

$F^*(G) = K_1 \times K_1{}^t$ with $K_1 \cong A_6$. Thus we may assume that t normalizes every component of $E(G)$.

Suppose that case (1) holds. By the L-Balance Theorem $E(C_G(t))$ lies in a unique component K of $E(G)$ and $C_{\langle K,t\rangle}(t)$ is isomorphic to $\mathbb{Z}_2 \times A_6$ or $\mathbb{Z}_2 \times M_{10}$. By inspection of the conclusions to [13], either $K \cong A_6$ or $K \cong U_4(3)$. In the former case we see first that $t \in O_2(G)$, then that $t \in Z(G)$, contrary to hypothesis. Thus $K \cong U_4(3)$. Also $C_G(\langle K, t\rangle) = \langle 1\rangle$. Thus $C_G(K) \subseteq O(G) = \langle 1\rangle$, whence $K = F^*(G)$.

Thus case (2) holds. Suppose that $K \trianglelefteq E(G)$, K quasi-simple. By inspection of the conclusions to [13], $m_2(K) \leqslant 2$. Also $E(G)$ has at most 4 components. Thus $G/C_G(E(G))$ does not involve $U_4(3)$. So we may assume that $G = C_G(E(G))$, i.e., $F^*(G)$ is a 2-group of sectional 2-rank at most 4. Thus $G/F^*(G)$ is isomorphic to a subgroup of $GL(4, 2)$. So G does not involve $U_4(3)$.

Finally, we need the following classification theorem.

THEOREM 5.4. *Let G be a finite group with $O(G) = \langle 1\rangle$ having a standard subgroup K with $K/Z(K)$ isomorphic to Ru, $Sz(2^{2n+1})$, $n \geqslant 1$, $Sp(4, 2^m)$, $U_4(2^m)$, $L_4(2^m)$, $U_5(2^m)$ or $L_5(2^m)$ for some $m \geqslant 3$. If $K/Z(K) \cong Ru$, assume that $|Z(K)| = 2$. Then one of the following holds:*

(1) *$K \subseteq K_1K_2 \trianglelefteq G$ with $K_i \trianglelefteq K_1K_2$ and $K_i \cong K$.*

(2) *$K \cong Sz(2^{2n+1})$, $\langle K^G\rangle \cong Sp(4, 2^{2n+1})$ and $|C_G(K)|_2 = 2$.*

(3) *$K \cong Sz(8)$ and $\langle K^G\rangle \cong Ru$.*

(4) *$K \cong Sp(4, 2^m)$ and $\langle K^G\rangle \cong Sp(4, 2^{2m})$, $U_4(2^m)$, $L_4(2^m)$, $U_5(2^m)$ or $L_5(2^m)$.*

(5) *$K \cong U_r(2^m)$ and $\langle K^G\rangle \cong L_r(2^{2m})$ for some $r \in \{4, 5\}$.*

(6) *$K \cong L_r(2^m)$ and $\langle K^G\rangle \cong L_r(2^{2m})$ for some $r \in \{4, 5\}$.*

Proof. The case $K \cong \hat{R}u$ is handled by Finkelstein in [10]. The case $K \cong Sz(2^{2n+1})$ is handled by Griess *et al.* [17] and Dempwolff [9]. The case $K \cong Sp(4, 2^m)$ is handled by Gomi [12]. The case $K \cong L_r(2^m)$ is handled by Seitz [28] and the case $K \cong U_r(2^m)$ is handled by Miyamoto [23].

We now fix G as a minimal counterexample to Theorem 1.1. Then G has a standard subgroup, $L \cong Ru$, by the results of Section 4. We let K be a standard subgroup of L, as given in Lemma 5.2(3).

LEMMA 5.5. *Hypothesis 3.1 holds and $|C_G(L)|_2 = 2$.*

Proof. As usual, $F^*(G)$ is simple. By Lemma 5.1(1), $m_2(K) = 3$. By Lemma 5.2(1)–(4), if $w \in I(L)$, then $O(C_L(w)) = \langle 1\rangle$ and any normal abelian 2-subgroup of $C_L(w)$ is elementary. Thus by Theorem 2.4, $|C_G(L)|_2 = 2$. Hypothesis 3.1(3) may be proved as in Lemma 4.18, using Theorem 5.4.

Let $R \in \mathrm{Syl}_2(K)$, $N = N_K(R)$. By Lemma 5.1(3), $C_{\mathrm{Aut}\,K}(N) = \langle 1\rangle$. Let

$C = C_L(N) \supseteq C_L(K) \cong U \cong E_4$. Then $U = C_C(u)$ for each $u \in U^\#$ by Lemma 5.2(3). Thus $U \in \mathrm{Syl}_2(C)$. Also, by Lemma 5.2(3), $U = N_C(U)$. Thus $U = C$.

By Lemma 5.2(2,3), if $K_1 \in \mathscr{L}(L)$ with $K_1 \cong K$, then K_1 is standard in L and $C_{\mathrm{Aut}\, L}(K_1) \cong E_4$. Now a Sylow 2-subgroup of $N_G(L) \cap C_G(K_1)$ is isomorphic to E_8. By Lemma 5.1(4), if α is an involutory automorphism of a central extension $\hat{K}_1$ of K_1, then either $m_2(C_{\hat{K}}(\alpha)) > 3$ or $C_{\hat{K}}(\alpha) \cong R$. In either case $C_{\hat{K}}(\alpha)$ is not isomorphic to a subgroup of E_8. This verifies Hypothesis 3.1.

Now by Lemma 3.3 we can pick $K < L \in \mathscr{L}^*(G)$ with the property that if $K < L_1 \in \mathscr{L}(G)$, then $L_1 \cong L$ and $L_1 \in \mathscr{L}^*(G)$. We fix such a pair (K, L) and let $\langle t \rangle \in \mathrm{Syl}_2(C_G(L))$, $U = C_L(K)$, $u \in U^\#$. We define $K(C_G(s))$ as in (3.4).

LEMMA 5.6. *$C_G(t) = L \times \langle t \rangle$ and $K = K(C_G(u))$.*

Proof. As G is a balanced group, $C_G(t) = L \times \langle t \rangle$. Suppose that $K \neq K(C_G(u))$. Then by the above remarks, $\langle K^{C_G(u)} \rangle = L_1 \cong Ru$ and $|C_G(L_1)|_2 = 2$. Let $E = \langle U, t \rangle$. It follows that $E = C_G(\langle K, t \rangle) = C_G(\langle K, u \rangle)$.

We argue that $N_G(E)$ is transitive on $E^\#$. $N_L(E)$ has orbits $\{t\}$, $U^\#$, $tU^\#$ on $E^\#$. $N_{L_1}(E)$ is transitive on $E^\# \cap L_1$. As $|U^\# \cap L_1| = 1$, $U^\# \sim tU^\#$ in $N_G(E)$. As $t^{N_{L_1}(E)} \neq \{t\}$, $N_G(E)$ is transitive on $E^\#$. Now

$$N_G(E) \cap C_G(t) = (K \times U)\langle r \rangle \times \langle t \rangle.$$

Clearly $K \trianglelefteq N_G(E)$. Let x be a 7-element of $N_G(E) - C_G(t)$. We may choose $x \in C_G(K)$.

Let $f \in C_K(r)$ of order 5. Then $C_G(\langle f, t \rangle) = M$ with $M \cong A_5$ and $U \in \mathrm{Syl}_2(M)$. As $x \in C_G(f)$ and x is transitive on $E^\#$, $C_G(f) = \langle f \rangle \times J$ with $J \cong J_1$ by Janko [22].

Now let $S_1 = C_S(u)$ where $\langle U, t \rangle \subseteq S \in \mathrm{Syl}_2(N_G(L))$. Let $A = \Omega_1(S_1)$ and let $v \in V = A \cap K$. As v is a square in K, $v \notin t^G$. $N_L(K)$ has orbits $U^\#$, $V^\#$ and $(A \cap L) - (U \cup V)$ on $(A \cap L)^\#$. If $g \in N_S(S_1) - S_1$, then $U^g \cap U = \langle 1 \rangle$. As $v^g \cap U = \varnothing$, $u^{N_L(A)} = (A \cap L) - V$. Thus

$$|u^{N_L(A)}| = 24 = |(tu)^{N_L(A)}|.$$

As $t \sim u \sim tu$ in $N_G(A)$, and $t \not\sim v$, $|t^{N_G(A)}| = 49$ or 56. As $7^3 \nmid |GL(6, 2)|$, $|t^{N_G(A)}| = 56$ and $t \sim tv$.

We now consider $C_G(r)$. We have

$$C_G(\langle t, r \rangle) \cong C_G(\langle tv, r \rangle) \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cdot M_{10}$$

and $v \in C_G(\langle t, r \rangle)'$. Thus $t \notin Z^*(C_G(r))$. Let $C = C_G(r)$, $\tilde{C} = C/O(C)$, $\hat{C} = C/\langle r \rangle$. By Lemma 5.3(1), $F^*(\tilde{C})$ is isomorphic either to $U_4(3)$ or to $A_6 \times A_6$. As $[L(\hat{C}), \hat{t}] = L(\hat{C})$ and $\hat{t}$ is fixed point free on $O(\hat{C})$, we have $[L(\hat{C}), O(\hat{C})] = \langle 1 \rangle$.

Thus $O(L(C)) \subseteq Z(L(C))$. Now $f \in L(C_G(\langle t, r\rangle)) \subseteq L(C)$. As $L(C)$ has an abelian Sylow 5-subgroup and $O^2(C) = L(C)\, C_C(L(C))$, f is central in a Sylow 5-subgroup of C. Thus $|C|_5 = 5^2$ and a Sylow 5-subgroup, F, of C is conjugate to a Sylow 5-subgroup of $C_L(f)$. In particular, $|f^G \cap F| = 20$. If $L(C)/Z(L(C)) \cong A_6 \times A_6$, then $F^{\#}$ contains at least 8 elements whose centralizers involve A_6. But $C_G(f)$ does not involve A_6. Thus $L(C)/Z(L(C)) \cong U_4(3)$ and the elements of $F^{\#} \cap O(C)$ are conjugate to a 5-central element, h, of L. But then $C_G(\langle h, t\rangle)/O_5(C_G(\langle h, t\rangle)) \cong \mathbb{Z}_2 \times Q_8$ and $C_G(h)$ involves $U_4(3)$, contrary to Lemma 5.3(2).

LEMMA 5.7. *Hypothesis* 3.8 *holds.*

Proof. By Lemmas 5.1(3) and 5.2(3), $C_L(u)$ is transitive on $\Omega_1(R)^{\#}$ where $R \in \mathrm{Syl}_2(K)$. Thus Condition 2 of Lemma 3.9 holds. By Lemma 5.2(3), Condition 4 of Lemma 3.10 holds, whence Condition 1 of Lemma 3.9 holds. By Lemma 5.2(1)–(4), Condition 1 of Lemma 3.11 holds. By Theorem 5.4, Condition 2 of Lemma 3.11 holds. Thus Condition 3 of Lemma 3.9 and hence, Hypothesis 3.8, holds.

We now reach a final contradiction. By Lemma 5.2(1)–(4), $Z(C_{\mathrm{Aut}\, L}(w)) = \langle w\rangle$ for all $w \in I(\mathrm{Aut}\, L)$. Thus Condition 1 of Lemma 3.12 holds. Also by Lemma 5.2(1)–(4), $u^L \cap C_L(w) \neq \varnothing$ for all $w \in I(\mathrm{Aut}\, L)$. Thus by Lemmas 3.12 and 3.13, $t \in Z^*(G)$, contrary to hypothesis.

6. THE PROOF OF THEOREM 1.3

In this section we complete the proof of Theorem 1.3. We fix G as a minimal counterexample to Theorem 1.3. By Theorem 4.2, G has a standard subgroup, L, with $L/O(L) \cong M(22)$ or $M(23)$ or $M(24)'$. As usual, $F^*(G)$ is simple and $|G{:}F^*(G)| \leqslant 2$. Our aim is to prove that $L \cong M(23)$, $|G{:}F^*(G)| = 2$ and $F^*(G)$ is $M(24)'$. We first collect the needed results about $U_6(2)$, $M(22)$, $M(23)$ and $M(24)'$.

LEMMA 6.1. *Let K be a perfect 2-fold covering group of $U_6(2)$ and let $K_0 = \mathrm{Aut}\, K \supseteq \bar{K} = \mathrm{Inn}\, K$.*

(1) *$K_0/\bar{K} \cong S_3$ and $m_2(K) \geqslant 3$.*

(2) *$\bar{K}$ has three classes of involutions. The centralizers of an involution in K is 2-constrained and core-free. If $\bar{v} \in I(\bar{K})$, then $C_{\bar{K}}(\bar{v})$ involves $\mathbb{Z}_3 \times \mathbb{Z}_3$. If $\bar{v} \in I(\bar{K})$ is 2-central, then $C_{\bar{K}}(\bar{v})$ involves $U_4(2)$.*

(3) *$K_0 - \bar{K}$ has two classes of involutions $g_1^{K_0}$ and $g_2^{K_0}$. $C_{\bar{K}}(g_1) \cong Sp(6, 2)$ and $C_{\bar{K}}(g_2)$ is an extension of E_{32} by S_6. In particular, $|C_{\bar{K}}(g_i)|_2 = 2^9 < |U_6(2)|_2$.*

Proof. The first half of (1) follows from Steinberg's general results. (See [8, Chap. 12]). Fact (2) is easily deduced from (6.2) of [5] and Fact (3) from (19.9) of [5]. In Lemma 6.2 we shall prove that K has a subgroup isomorphic to $Sp(6, 2)$. Thus $m_2(K) \geqslant 6$ and (1) is proved.

LEMMA 6.2. *Let L_1 be a quasi-simple group with $L_1/O(L_1) \cong M(22)$. Let $L_0 = \operatorname{Aut} L_1 \supseteq \bar{L}_1 = \operatorname{Inn} L_1$ and let $\hat{L}_1$ be a perfect 2-fold covering group of $\bar{L}_1$.*

(1) *$|L_0 : \bar{L}_1| = 2$ and the Schur multiplier of $\bar{L}_1$ has order 6. Also $m_2(L_1) \geqslant 3$.*

(2) *$\bar{L}_1$ has three classes of involutions: $u^{\bar{L}_1}$, $v^{\bar{L}_1}$, $w^{\bar{L}_1}$.*

(3) *u is not 2-central. $C_{\bar{L}_1}(u)$ is a perfect 2-fold covering group of $U_6(2)$.*

(4) *v is 2-central. $O_2(C_L(v)) \subseteq C_L(v)^{(\infty)}$ and $O_2(C_L(v)) = \langle u \rangle \times E$ with E extra-special of order 2^9 and $\langle v \rangle = Z(E) \cdot C_L(v)'/O_2(C_L(v)) \cong U_4(2)$.*

(5) *w is not 2-central. $|C_L(w)| = 2^{16} \cdot 3^3$.*

(6) *$L_0 - \bar{L}_1$ has three classes of involutions: x^{L_0}, y^{L_0}, z^{L_0}.*

(7) *$C_{\bar{L}_1}(x) \cong O^+(8, 2)$. $C_{\bar{L}_1}(y) \cong \mathbb{Z}_2 \times Sp(6, 2)$. $C_{\bar{L}_1}(z)$ is an extension of an elementary group of order 64 by $\Omega^-(6, 2)$.*

(8) *If $s \in I(L_0)$, then $u^{\bar{L}_1} \cap C_{\bar{L}_1}(s) \neq \varnothing$.*

(9) *If $s \in I(L_0 - \bar{L}_1)$, then $|C_{\bar{L}_1}(s)|_2 \leqslant 2^{16} < |M(22)|_2$.*

Proof. The Schur multiplier of $M(22)$ is determined in [16]. The information in (2)–(8) appears in (13.8) of [6]. Now $C_{L_0}(u) = C_{\bar{L}_1}(u)\langle s \rangle$ for some involution $s \in L_0 - \bar{L}_1$ by (13.8) of [6]. We may assume that $C_{\bar{L}_1}(\langle u, s \rangle)/\langle u \rangle \cong Sp(6, 2)$. As $C_{\bar{L}_1}(r)$ does not involve $Sp\hat{}(6, 2)$ for all $r \in I(L_0 - \bar{L}_1)$, we conclude that $C_{\bar{L}_1}(\langle u, s \rangle) \cong \mathbb{Z}_2 \times Sp(6, 2)$. The inverse image in $\hat{L}_1$ of $E(C_{\bar{L}_1}(\langle u, s \rangle))$ is $\mathbb{Z}_2 \times Sp(6, 2)$ or $Sp\hat{}(6, 2)$. In either case the 2-rank is at least 4. This proves (1). Statement (9) is immediate from (7).

LEMMA 6.3. *Let $L = M(23)$.*

(1) *$L = \operatorname{Aut} L$ has 3 classes of involutions: u^L, v^L, w^L. All are 2-central.*

(2) *$C_L(u)$ is a perfect 2-fold covering group of $M(22)$.*

(3) *$O_2(C_L(v)) \subseteq O^2(C_L(v))$. $O_2(C_L(v)) = A \times E$ with $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, E extra-special of order 2^9 and $\langle v \rangle = Z(E)$. $O^2(C_L(v))/O_2(C_L(v)) \cong \mathbb{Z}_3 \times U_4(2)$.*

(4) *$|C_L(w) : C_L(w)'| = 2$ and $C_L(w)' = F^*(C_L(w))$ is a perfect 4-fold covering group of $U_6(2)$.*

(5) *Let $T \in \operatorname{Syl}_2(L)$. T has a unique elementary subgroup, W, of order 2^{11} and $N_L(T)/T \cong M_{23}$. Also $|u^L \cap W| = 23$.*

Proof. Facts (1), (2) and (4) are in (13.7) of [6]. Fact (3) is in [27]. Fact (5) is in [21].

LEMMA 6.4. Let $M = M(24)'$. $M_0 = \mathrm{Aut}\, M$. The following properties hold:

(1) $|M_0 : M| \leqslant 2$ and the Schur multiplier of M has order 1 or 3.

(2) M has two classes of involutions: u^M, v^M.

(3) u is not 2-central. $C_M(u)'$ is a perfect 2-fold covering group of $M(22)$. $C_M(u)/\langle u\rangle \cong \mathrm{Aut}\, M(22)$. $C_{M_0}(u) = (K \times \langle u_1\rangle)\langle x\rangle$ where $K = C_M(u)'$, $\langle u_1, x\rangle \cong D_8$ with $\langle u\rangle = Z(\langle u_1, x\rangle)$.

(4) v is 2-central. $O_2(C_M(v)) = O_2(C_{M_0}(v))$ is extra-special of order 2^{13}. $O^2(C_{M_0}(v))/O_2(C_{M_0}(v))$ is a perfect 3-fold covering group of $U_4(3)$, acting faithfully on $O_2(C_{M_0}(v))/\langle v\rangle$.

(5) $M_0 - M$ has two classes: $u_1{}^M$, w^M.

(6) $C_M(u_1) \cong M(23)$ and $C_M(u_1) \supseteq C_M(u)'$ for u_1 as in (3).

(7) $C_M(w)^{(\infty)}$ is a perfect 4-fold covering group of $U_6(2)$ and $C_M(w)/O_2(C_M(w)) \cong \mathrm{Aut}\, U_6(2)$ permuting transitively $O_2(C_M(w))^{\#}$. There exists $w_1 \in w^L \cap C_{M_0}(\langle u, u_1\rangle)$.

(8) $|M| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$.

Proof. The precise structure of $C_{M_0}(v)$ is described in [27]. The Schur multiplier is shown to be 1 or 3 in [15]. The other data may be found in (13.1) and (13.6) of [6].

LEMMA 6.5. *Let H be a finite group with a normal self-centralizing elementary 2-subgroup E of order 2^{12} having an involution e with $|e^H| = 24$ and $C_H(e) = H_1 \times \langle e\rangle$, $H_1 \cap E = E_1$ and $H_1/E_1 \cong M_{23}$. Then $H/E \cong M_{24}$, $|H : H'| = 2$, $e \notin H'$ and E is the unique subgroup of H which is elementary of order 2^{12}.*

Proof. As M_{24} is the unique transitive extension of M_{23} of degree 24, $H/E \cong M_{24}$. By (8.4) of [4], E is indecomposable with irreducible submodule E_1. As M_{24} has trivial multiplier, $e \notin H'$. Also, (8.4) of [4] proves that E_1 is the unique elementary subgroup of H' of order 2^{11}. The last statement of this lemma follows trivially from this.

LEMMA 6.6. *Suppose that $K \in \mathscr{L}(G)$ with $K/Z(K)$ isomorphic to $U_6(2)$, $M(22)$, $M(23)$ or $M(24)'$. If $K/Z(K) \cong U_6(2)$, assume that $|Z(K)|$ is even. Suppose that $K \lll L \in \mathscr{L}(G)$ and $L/Z(L) \cong K/Z(K)$. Then one of the following holds*:

(1) $K/Z(K) \cong U_6(2)$, $L/Z(L) \cong M(22)$.

(2) $K/Z(K) \cong U_6(2)$ *or* $M(22)$, $L \cong M(23)$.

(3) $K/Z(K) \cong U_6(2)$, $M(22)$ *or* $M(23)$, $L/Z(L) \cong M(24)'$.

Proof. As in the proof of Lemma 4.19, we may reduce to that case that $\bar{K} = KZ(L)/Z(L)$ is standard in $N_G(L)/Z(L)$. By the minimality of G, we know $|Z(K)|$ is even. Then by Hypothesis 1.2, the Corollary holds.

We now pick a standard subgroup $L \in \mathscr{L}^*(G)$ satisfying:

(1) $|Z(L)|$ is odd, $L/Z(L) \cong M(22)$ or $M(23)$ or $L/Z(L) \cong M(24)'$.

(2) $|L|$ is maximal subject to (1).

We let K be a standard subgroup of $\langle L, u\rangle$ for some $u \in I(N_G(L) - C_G(L))$ with $K/Z(K) \cong U_6(2)$, $M(22)$ or $M(23)$ and with $|K|$ maximal subject to this. We pick $u \in K$, if possible. By Lemmas 6.2, 6.3 and 6.4, either $u \in K$ or $K \cong M(23)$, $L/Z(L) \cong M(24)'$. For each possibility for K, we define a set $\mathscr{L}$ as follows:

(1) If $K/Z(K) \cong U_6(2)$, $\mathscr{L} = \{L_0, L_1\}$ where $L_i/Z(L_i) \cong M(22)$, $|Z(L_0)| = 1$ and $|Z(L_1)| = 3$.

(2) If $K/Z(K) \cong M(22)$, then $|Z(K)| = 2$ and $\mathscr{L} = \{L_2, L_3, L_4\}$ with $L_2 \cong M(23), L_i/Z(L_i) \cong M(24)'$ for $i = 3, 4$, $|Z(L_3)| = 1$ and $|Z(L_4)| = 3$.

(3) If $K \cong M(23)$, then $\mathscr{L} = \{L_3, L_4\}$ with L_i as in (2).

Finally, in Lemma 6.11, we shall need the following transfer lemma of Yoshida [33].

LEMMA 6.7. *Let G be a finite group, Q a weakly closed abelian 2-subgroup of G, $N = N_G(Q)$. Suppose there exists $x \in I(Q)$ and $N_0 \trianglelefteq N$ of index 2 with $x \notin N_0$. Then $x \notin O^2(G)$.*

We may now begin the proof of Theorem 1.3 for the $M(i)$ cases.

LEMMA 6.8. *Conditions* (1), (2) *and* (4) *of Hypothesis* 3.1 *hold for* $(K, \mathscr{L})$ *as above.*

Proof. By Lemmas 6.1(1) and 6.2(1), $m_2(K) \geqslant 3$ in cases (1) and (2). As a 2-fold covering group of $M(22)$ lies in $M(23)$, $m_2(K) \geqslant 3$ in case (3) as well. By the minimal choice of G, if $L \cong L_3$ or L_4, then $L \in \mathscr{L}^*(G)$. If $L \ncong L_3$ or L_4, then by maximal choice of L, only L_0, L_1 and L_2 can occur in $\mathscr{L}(G)$. In this case, if $L \cong L_2$, then by induction, $L \in \mathscr{L}^*(G)$. If $L \ncong L_2$, L_3 or L_4, then only L_0 and can occur and by Hypothesis 1.2, $L \in \mathscr{L}^*(G)$. Now Hypothesis 3.1(1) and (2) hold.

By Lemmas 6.2(3) and 6.3(2) and 6.4(3,6), $N_G(L) \cap C_G(K)$ is 2-nilpotent with Sylow 2-subgroup $\langle t_0, u\rangle$ where $\langle t_0\rangle$ is a cyclic Sylow 2-group of $C_G(L)$, $\langle u\rangle \cong C_{\mathrm{Aut}\, L}(K) \cong \mathbb{Z}_2$. Now if K_2 is a perfect central extension of K and α is an involutory automorphism of K_2, then $O(C_{K_2}(\alpha)) = \langle 1\rangle$ and $C_{K_2}(\alpha)$ involves $\mathbb{Z}_3$ by Lemmas 6.1, 6.2, 6.3 and 6.4. Thus $C_{K_2}(\alpha)$ is not isomorphic to a subgroup of $N_G(L) \cap C_G(K)$. Thus Condition (4b) holds.

If $K \not\simeq M(23)$, we may take $N = K_1$ to satisfy Condition (4a) by the above. If $K \simeq M(23)$, take $N = C_K(u_1)$ with $N/\langle u_1\rangle \simeq M(22)$. N exists by Lemmas 6.3(2) and 6.4, $C_G(N) \cap N_G(L)$ is 2-nilpotent and Condition (4a) holds in this case as well.

LEMMA 6.9. *Condition* (3b) *of Hypothesis* 3.1 *holds and either Condition* (3a) *holds or* $K \simeq M(23)$.

Proof. Condition (3b) is proved in Lemma 6.6. If $K \not\simeq M(23)$, then Hypothesis 1.2 implies Condition (3a).

LEMMA 6.10. *Let* $t \in I(C_G(L))$, $u \in I(N_G(L) \cap C_G(K)) - I(C_G(L))$, *with* $u \in I(K)$, *if possible. Let* $L_u = L(C_G(u))$, $L_{ut} = L(C_G(ut))$. *Then for suitable choice of* (K, L) *and* u, *one of the following holds*:

(1) $K = K(C_G(u))$.

(2) $L/O(L)$, $L_u/O(L_u)$ *and* $L_{ut}/O(L_{ut})$ *are all isomorphic to* $M(24)'$. $K \simeq M(23)$.

Proof. Suppose $K \not\simeq M(23)$. Then by Lemmas 3.3, 6.8 and 6.9, we may pick (K, L) to be a K-extremal pair. Moreover, $u \in K$. Thus $K = K(C_G(u))$ by Lemma 3.6.

Suppose $K \simeq M(23)$. By Lemma 6.9, either Hypothesis 3.1 holds for (K, L) or $K_1 \in \mathscr{L}^*(G)$ for some $K_1 \simeq K$. By the proof of Lemma 3.3, either the conclusion of Lemma 3.3 holds or there exists a chain

$$K = K_0 < K_1 < \cdots < K_n < K_{n+1} = F^*(G)$$

satisfying the conditions of Corollary 2.3 with $K_n/Z(K_n)$ not isomorphic to $M(24)'$. Thus $K_n \simeq M(23)$. If $n \geq 1$, then by Corollary 2.3(5), $N_G(K_n) \supseteq K_n \times (K_n)^s$ with s a 2-element, $K_{n-1} = C_{K_n \times (K_n)^s}(s)$. But $[K_n, (K_n)^g] \neq \langle 1\rangle$ for all $g \in G$ by Theorem 2.2. Thus $n = 0$. But then K is standard in G, not the case. Thus Lemma 3.3 holds. Now clearly either Conclusion (1) or (2) holds.

We now fix (K, L, u) satisfying one of the conclusions of Lemma 6.10. We let t, L_u and L_{ut} be as in Lemma 6.10.

LEMMA 6.11. $K/Z(K) \not\simeq U_6(2)$.

Proof. Suppose that $K/Z(K) \simeq U_6(2)$. We prove first that Hypothesis 3.8 holds. Let $S \in \mathrm{Syl}_2(C_L(u))$. Then Condition 3 of Lemma 3.10 holds trivially by Lemma 6.2(3). Thus 3.8(1) holds. Let $S \subsetneq T \in \mathrm{Syl}_2(L)$. As $|T : S| = 2$, $\Omega_1(Z(S)) = \langle u, v\rangle$ with $\langle v\rangle = \Omega_1(Z(T))$ by Lemma 6.2(4). Let $S_1 \in \mathrm{Syl}_2(M(23))$. Then $Z(S_1)$ contains involutions u_1, v_1, w_1 and $C_{M(23)}(v_1)$ does not involve $U_6(2)$, by 6.3(3). It follows by 6.3(4), from the structure of $C_{M(23)}(w_1)$ that $v_1 \in E(C_{M(23)}(w_1)) - Z(E(C_{M(23)}(w_1)))$. As $C_{M(23)}(w_1)/O_2(C_{M(23)}(w_1))$ contains a Sylow 2-subgroup of Aut $U_6(2)$, it follows in our original context that $v \not\sim uv$

in $N_G(K)$. Thus 3.8(2) holds. Finally, the conditions of Lemma 3.11 hold by Lemma 6.2 and Corollary 6.6. Thus we have $C_G(u) \subseteq N_G(L)$. Next, by Hypothesis 1.2, $u \in Z(E(C_G(ut)))$ or $E(C_G(ut)) = L_1 \cong L$.

By Lemmas 3.13 and 6.2(8), Condition 3 of Lemma 3.12 holds. Let $t_1 = t^g \in C_G(\langle u, t\rangle) - \langle t\rangle$. Then $C_G(\langle u, t, t_1\rangle) = C_G(\langle u, t\rangle)$. By the proof of Lemma 3.12, we may assume that $t_1 \neq tu$. Let $C = C_G(t_1)$. Then $Z(C_C(u))$ has 2-rank at least 3. It follows that either $C_C(u)$ is solvable with $|C_C(u)|_2 = |C_G(u)|_2$ or $E(C_C(u)) \cong Sp(6, 2)$. In the former case t_1 is 2-central in $C_G(\langle u, t\rangle)$. But then by Lemma 6.1(2), $C_G(\langle u, t, t_1\rangle)$ involves $U_4(2)$, a contradiction. Thus $E(C_C(u)) \cong Sp(6, 2)$. As both u and t induce outer automorphisms of L^g, ut induces an inner automorphism of L^g and $E(C_{L^g}(ut)) \neq \langle 1\rangle$. Thus $E(C_{L^g}(ut)) \cong U_6(2)$ and $u \notin Z(E(C_G(ut)))$. Thus $L_1 = E(C_G(ut)) \cong L$. Now t and t_1 both act on L_1 with $E(C_{L_1}(t)) \cong U_6(2) \cong E(C_{L_1}(t_1))$. Thus both t and t_1 induce inner automorphisms on L_1. But $E(C_{L_1}(\langle t, t_1\rangle)) \cong Sp(6, 2)$, a contradiction.

LEMMA 6.12. *$L/O(L) \cong M(24)'$ type.*

Proof. Suppose that $L \cong M(23)$. As $u \in K$, Hypothesis 3.5 holds. The rest of the general argument does not apply but we may follow the spirit of the argument. By Glauberman's Z^*-Theorem, $t^G \cap C_G(t) \neq \{t\}$. As $t \notin C_G(t)'$, $t^G \cap L = \varnothing$. By Lemma 6.3, if $s \in I(L)$, then any abelian normal subgroup of $C_L(s)$ is an elementary 2-group. Thus $\langle t\rangle \in \mathrm{Syl}_2(C_G(L))$ by Theorme 2.4(2). Let $T_0 \times \langle t\rangle = T_1 \in \mathrm{Syl}_2(C_G(t))$ with $u \in Z(T_0)$. Suppose that $t \sim tx$ with $x \in Z(T_0) - \langle u\rangle$. Let $tx = t^h$.

By Glauberman's Z^*-Theorem, there exists $u_1 \in (u^L \cap K) - \{u\}$. Let $g \in C_K(K)$. Then g normalizes $\langle K, K(C_G(u_1))\rangle = \langle K, K(C_L(u_1))\rangle = L$, by Hypothesis 1.2. Thus $KC_G(K) \subseteq N_G(L)$. As $|C_G(u): KC_G(K)| \leqslant 2$, $|C_G(u): C_G(u) \cap N_G(L)| \leqslant 2$.

By Lemma 6.3, $C_L(\langle u, x\rangle) = (A \times E)J$, where $A \cong E_4$, E is extra-special of order 2^9 and $J \cong SO(5, 3)$. Clearly $|C_{L^h}(u): C_L(u) \cap L^h| = 2^n$ for some $n \geqslant 0$. But this is impossible by inspection of the possible centralizers listed in Lemma 6.3.

Thus we have shown that $t^G \cap Z(T_1) = \{t, tu\}$. It follows that $t^G \cap T_1 = \{t\} \cup \{(tu)^L \cap T_1\}$. By Lemma 6.3(5), T_1 contains a unique elementary abelian subgroup, W, of order 2^{12}. Moreover $N_{L\times\langle t\rangle}(W)/W \cong M_{23}$ and $|(tu)^L \cap W| = 23$. Thus by Lemma 6.5, $N_G(W)/C_G(W) \cong M_{24}$. Let $T_2 \in \mathrm{Syl}_2(N_G(W))$. By Lemma 6.5, W is weakly closed in $N_G(W)$ with respect to G. Thus $T_2 \in \mathrm{Syl}_2(G)$. Also, by Lemma 6.5, $N_G(W)$ has a normal subgroup N_0 of index 2 with $t \notin N_0$. By Lemma 6.7, $t \notin O^2(G)$. Thus $\langle u\rangle \in \mathrm{Syl}_2(C_G(K))$. Hence K is standard in $O^2(G)$. Then, by Hypothesis 1.2, $O^2(G) \cong M(24)'$. But then G is not a counterexample to Theorem 1.3, contrary to hypothesis.

LEMMA 6.13. *$K \cong M(23)$.*

Proof. Suppose that $K/\langle u\rangle \cong M(22)$ and $L/O(L) \cong M(24)'$. As $u \in K$, Hypothesis 3.5 holds. As K is standard in $N_G(L)$, $N_G(L) = LC_G(L)$. Let $S \in \mathrm{Syl}_2(C_L(u))$, $R = S \cap K$. Then as u is not 2-central by Lemma 6.4(3), $S \notin \mathrm{Syl}_2(L)$. Also by Lemma 6.2(9), $Z(S) \subseteq R$. By Lemma 6.2(9) and Definition 6.4(3), $SK/\langle u\rangle \cong \mathrm{Aut}\, M(22)$. Thus Conditions (1) and (2) of Lemma 3.9 hold.

Let $uv \in u^G \cap Z(R)$ with $uv \neq u$ and suppose that $L_0 = \langle K, K(C_G(uv))\rangle \neq L$. Then by Hypothesis 1.2, $L_0 \cong M(23)$ and $R \in \mathrm{Syl}_2(L_0)$. But S normalizes $\langle u, uv\rangle$, hence L_0, and $|C_L(L_0)|$ is odd. This is impossible as $L_0 = \mathrm{Aut}\, L_0$ by Lemma 6.3(1). Thus $L_0 = L$ and Hypothesis 3.8 holds.

By Lemma 6.4, if $s \in I(L)$, $Z(C_L(s)) = \langle s\rangle$. Thus Condition 1 of Lemma 3.12 holds. Clearly, by Lemma 6.4, Condition 3 also holds and we have a contradiction by Lemma 3.12.

We fix our notation for the final case. We have $L \in \mathscr{L}^*(G)$ with $L/O(L) \cong M(24)'$. By Lemma 6.4 and Theorem 2.4(2), $|C_G(L)|_2 = 2$. We let $\langle t\rangle \in \mathrm{Syl}_2(C_G(L))$, $u \in N_G(L)$ with $K = E(C_L(u)) \cong M(23)$. We let $L_u = L(C_G(u))$, $L_{ut} = L(C_G(ut))$.

LEMMA 6.14. *$L_u/O(L_u)$ and $L_{ut}/O(L_{ut})$ are isomorphic to $M(24)'$.*

Proof. If not, then by Lemma 6.10, $K = K(C_G(u))$. As $|M(23)|_2 < |L|_2$, u is not 2-central. By Lemma 6.3(1), $M(23) = \mathrm{Aut}\, M(23)$. Thus Conditions (1) and (2) of Lemma 3.9 hold. Let $v \in I(K)$ with $uv \in u^L$ and let $H = \langle K, K(C_G(uv))\rangle$. By induction, $H/O(H) \cong M(24)'$, whence $|H/O(H)| = |L/O(L)|$ by Lemma 6.4(8). Thus $H = L$. Hence Hypothesis 3.8 holds, i.e., $C_G(u) \subseteq N_G(L)$. By Lemma 6.4 and Lemma 3.13, there exists $t_1 \in t^G \cap C_G(\langle u, t\rangle)$ with $t_1 \notin \{t, tu\}$. Now $O_2(Z(C_{\langle L,s\rangle}(s))) = \langle s\rangle$ for all $s \in I(\mathrm{Aut}\, L)$ by Lemma 6.4. Thus Lemma 3.12 yields a contradiction.

We now let $u_0 \in I(C_L(u))$ with $K_0 = E(C_L(u_0)) \cong M(22)$. By Hypothesis 1.2, either $K_0 \trianglelefteq E(C_G(u_0))$ or $K_1K_1{}^t \trianglelefteq E(C_G(u_0))$ with $K_0 = C_{K_1K_1{}^t}(t)'$. But $\langle t, u\rangle$ normalizes $K_1K_1{}^t$ and

$$K_0 = C_{K_1K_1{}^t}(t)' = C_{K_1K_1{}^t}(u)' = C_{K_1K_1{}^t}(ut)'.$$

Thus $K_0 \trianglelefteq E(C_G(u_0))$. As $C_G(\langle K_0, t\rangle)$ has Sylow 2-subgroup $\langle u, u_0, t\rangle$, K_0 is weakly closed in $C_G(u_0)$ with respect to G.

We have verified in Lemma 6.13 that Hypothesis 3.8 holds for the pair (K_0, L). Thus $C_G(u_0) \subseteq N_G(L)$. Thus

$$C_G\langle(u, u_0\rangle)/O(C_G(\langle u, u_0\rangle)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \hat{M}(22).$$

Now $C_G(u)/O(C_G(u)) = \bar{C} = \langle \bar{u}\rangle \times \bar{L}_u\langle \bar{t}\rangle$ with $\bar{L}_u \cong M(24)'$. By Lemma 6.4, $\bar{C}$ has no involutions with fixed points isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \hat{M}(22)$, a contradiction.

This completes the proof of Theorem 1.3.

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